Problems for 3D Heat and Wave Equations

18.303 Linear Partial Differential Equations

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1 Problem 1

A rectangular metal plate with sides of lengths L, H and insulated faces is heated to a uniform temperature of u_0 degrees Celsius and allowed to cool with its edges maintained at 0^o C. You may use dimensional coordinates, with PDE

$$u_t = \kappa \nabla^2 u, \qquad 0 \leq x \leq L, \qquad 0 \leq y \leq H.$$

(i) Find the smallest eigen-value λ. For large time t, the temperature is given approximately by the term with $e^{-\lambda\kappa t}$. Show that this term is

$$u(x, y, t) = A \exp\left(-\pi^2 \left(\frac{1}{L^2} + \frac{1}{H^2}\right) \kappa t\right) \sin\left(\frac{\pi x}{L}\right) \sin\left(\frac{\pi y}{H}\right).$$

Find the value of A. For fixed $t = t_0 \gg 0$, sketch the level curves $u =$ constant in the xy-plane.

(ii) Of all rectangular plates of equal area, which will cool the slowest? Hint: for each type of plate, the smallest eigenvalue gives the rate of cooling.

(iii) Will a square plate, side length L, cool more or less rapidly than a rod of length L, with insulated sides, and with ends maintained at 0^o C? You may use the results we derived in class for the rod, without derivation.

2 Problem 2

A rectangular metal plate with sides of lengths L, H and insulated faces has two parallel sides maintained at 0^o C, one side at 100^o C, and one side insulated.

(i) Find the equilibrium (steady-state) temperature $u_E(x, y)$ of the plate.

(ii) If $H = 2L$, approximate the temperature at

(a) the hottest point on the insulated edge, and

(b) at the center of the plate (use first-term approx).

Find an upper bound for the error in (a) and (b) by getting an upper bound on the ratio of the second to the first term on the insulated edge.

Bonus: use a symmetry argument to find the exact answer for (a).

(iii) Sketch typical isothermal curves and heat flow lines (the orthogonal trajectories). Where is the temperature gradient ∇u equal to zero?

(iv) Consider a square plate of side length L, with one side at 100^o C, an adjacent edge at 0^o C, and the other two edges insulated. Find the steady-state temperature at points A (center of the edge opposite the side at 0^o) and B (corner joining insulated sides) without doing any more calculation, i.e., use your solution to (ii) and symmetry. Hint: lines of symmetry in the original plate (with $H = 2L$) are heat flow lines, and can effectively divide the square into smaller parts, where the lines of symmetry are insulating boundaries.

3 Problem 3

Consider the eigenvalue problem for the Laplacian

$$\begin{aligned} \nabla^2 v + \lambda v &= 0 \quad \text{in} \quad D, \\ v &= 0 \quad \text{on} \quad \partial D, \end{aligned}$$

where $D = \{(x, y, z) : x^2 + y^2 + z^2 \leq 1\} \subset \mathbb{R}^3$ is a sphere of radius 1.

(i) If v is a pure radial function, $v = R(r)$, show that the eigenvalue problem reduces to

$$\frac{d^2}{dr^2}(rR) + \lambda r R = 0,$$

$$R(1) = 0, \qquad R(0) \text{ bounded}$$

Show that the radial eigen-functions and eigenvalues are

$$R_n(r) = g(n\pi r), \qquad n = 1, 2, 3, ...$$

where

$$g(x) = \begin{cases} \frac{\sin x}{x}, & x > 0 \\ 1, & x = 0 \end{cases}$$

Sketch the graphs of the first two eigen-functions.

You may use the fact that for $v = R(r)$, the Laplacian is

$$\nabla^2 R(r) = \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{dR}{dr} \right)$$

(ii) A solid sphere of dimensionless radius 1 is heated uniformly to a temperature of u_0 degrees Celsius, placed in ice at 0^o, and allowed to cool. Show that the temperature at the center for $t > 0$ is given by

$$u(0,t) = 2u_0 \sum_{n=1}^{\infty} (-1)^{n-1} \exp\left(-n^2 \pi^2 t\right)$$

Hint: use the dimensionless heat equation $u_t = \nabla^2 u$. What are the BCs? What is the IC?

Compare the central temperature with the temperature at the center of a rod of scaled length 2, and the same initial temperature. You may use the results we derived in class - you'll need to rescale the spatial coordinate x via $\hat{x} = 2x$ to make the scaled rod length 2, rather than 1.

4 Problem 4

Find the eigenvalue λ and corresponding eigen-function v for the isosceles right triangle; v and λ satisfy

$$\begin{aligned} \nabla^2 v + \lambda v &= 0 \quad \text{in} \quad D, \\ v &= 0 \quad \text{on} \quad \partial D, \end{aligned}$$

where $D = \{(x,y) : 0 < y < x, \quad 0 < x < 1\}$.

Hint: combine the eigen-functions on the square $D_{sq} = \{(x,y) : 0 < x < 1, 0 < y < 1\}$ to obtain an eigen-function on D that is positive on D. We know that the first eigen-function can be characterized (up to a non-zero multiplicative constant) as the eigen-function that is of one sign.

5 Problem 5

Consider the boundary value problem

$$\begin{aligned}
\nabla^2 v &= 0, \qquad 0 < x < 1, \qquad 0 < y < 1 \\
v(0, y) &= 0, \qquad v(1, y) = 100, \qquad 0 < y < 1 \\
v(x, 0) &= 100, \qquad v(x, 1) = 0, \qquad 0 < x < 1.
\end{aligned}$$

Give a symmetry argument to show that $v(x, x) = 50$ for $0 < x < 1$. Sketch the level curves of v.