Solutions for Problem Set 2 : Variations of the Basic Heat Problem

18.303 Linear Partial Differential Equations

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1 Problem 2

Consider the non-homogeneous heat problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + b; \qquad u(0,t) = 0 = u(1,t); \qquad u(x,0) = 0 \tag{1}$$

where $t > 0$, $0 < x < 1$ and b is constant.

a. Find the equilibrium solution $u_E(x)$.

Solution: The equilibrium solution $u_E(x)$ satisfies the PDE (1) and BCs,

$$0 = u_E'' + b; \qquad u_E(0) = 0 = u_E(1).$$

The solution is

$$u_E(x) = \frac{b}{2} x (1 - x)$$

b. Transform the heat problem (1) into a standard homogeneous heat problem for a temperature function $v(x,t)$.

Solution: Let

$$v(x,t) = u(x,t) - u_E(x)$$

and substitute into (1) to obtain

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2}; \qquad v(0,t) = 0 = v(1,t); \qquad v(x,0) = -u_E(x) \tag{2}$$

This is the basic Heat Problem.

c. Show that after a large time, the solution of the heat problem (1) is approximated by

$$u(x,t) \approx u_E(x) + Ce^{-\pi^2 t}\sin(\pi x).$$

Find C and comment on the physical significance of its sign. Illustrate the solution qualitatively by sketching typical temperature profiles t = constant and the central amplitude profile $x = 1/2$.

Solution: The solution to the basic heat problem (2) is

$$v(x,t) = \sum_{n=1}^{\infty} B_n \sin(n\pi x)\, e^{-n^2\pi^2 t}$$

where

$$B_n = 2\int_0^1 (-u_E(x))\sin(n\pi x)\,dx = -b\int_0^1 x(1-x)\sin(n\pi x)\,dx = \frac{2b}{\pi^3 n^3}(1-\cos\pi n)$$

Thus

$$B_n = \begin{cases} -\frac{4b}{\pi^3 n^3}, & n \text{ odd}, \\ 0, & n \text{ even}. \end{cases}$$

Hence

$$v(x,t) = -\frac{4b}{\pi^3}\sum_{m=1}^{\infty} \frac{\sin((2m-1)\pi x)}{(2m-1)^3} e^{-(2m-1)^2\pi^2 t}$$

After a large time, the first term dominates, so that

$$v(x,t) \approx B_1 \sin(\pi x)\, e^{-\pi^2 t} = -\frac{4b}{\pi^3}\sin(\pi x)\, e^{-\pi^2 t}$$

and

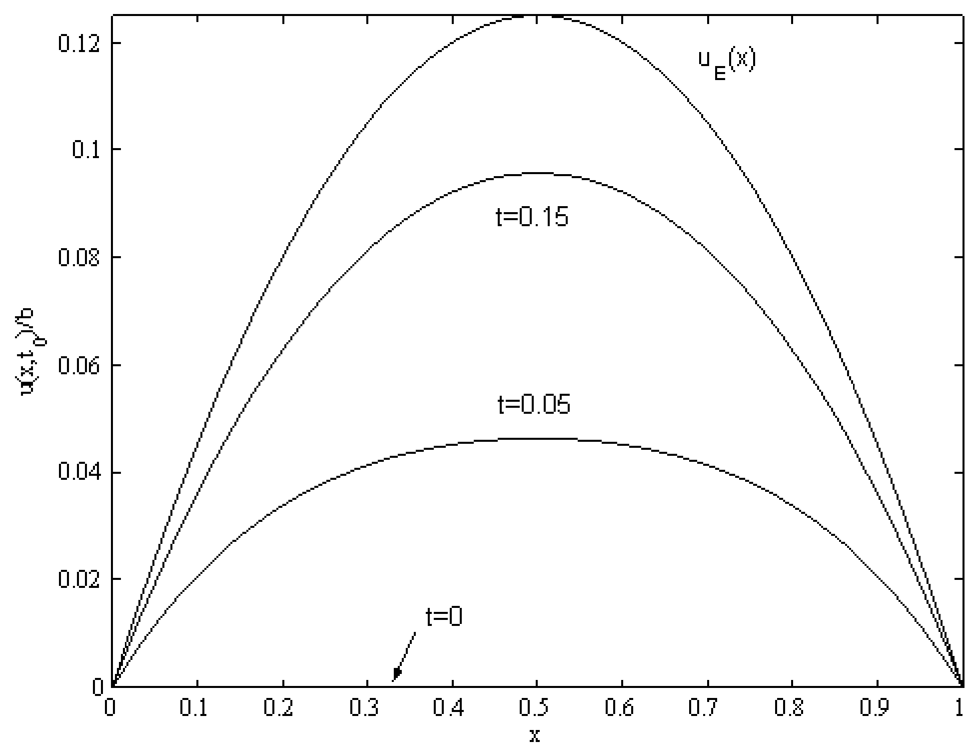
$$u(x,t) \approx u_E(x) - \frac{4b}{\pi^3} e^{-\pi^2 t}\sin(\pi x)$$

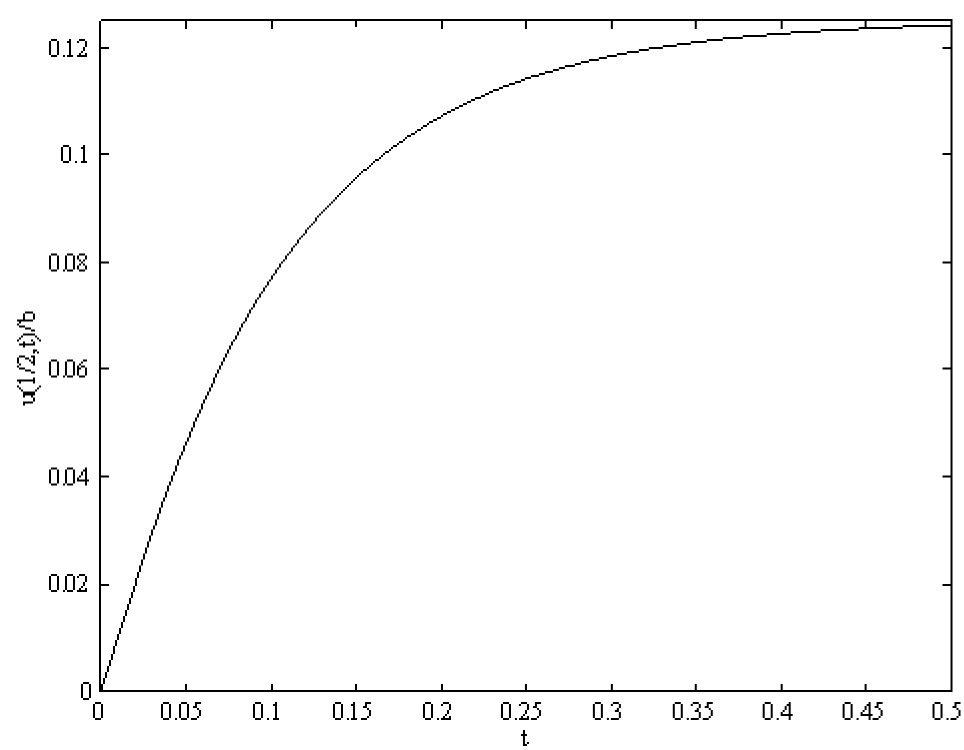
Thus $C = 4b/\pi^3$.

For plots, note that $u_E(x)$ is an upside-down parabola whose vertex is at $(1/2, -b/8)$ in the ux-plane. Also,

$$u\left(\frac{1}{2}, t\right) \approx u_E\left(\frac{1}{2}\right) - \frac{4b}{\pi^3} e^{-\pi^2 t}\sin\left(\frac{\pi}{2}\right) = \frac{b}{8} - \frac{4b}{\pi^3} e^{-\pi^2 t}.$$

The plots are below.


u_E(x)
t=0.15
t=0.05
t=0
u(x,t_0)/b
x


u(1/2,t)/b
t

2 Problem 4

Show that if u is a solution of the generalized heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + b\frac{\partial u}{\partial x} + cu + g(x,t) \tag{3}$$

where b, c are constants, then

$$v(x,t) = e^{\alpha x + \beta t} u(x,t) \tag{4}$$

satisfies the standard heat equation

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + h(x,t)$$

for suitable choices of the constants α, β and function $h(x,t)$. In this way, more complicated heat problems can be simplified.

Solution: Re-writing (4) for $u(x,t)$ gives

$$u(x,t) = e^{-\alpha x - \beta t} v(x,t) \tag{5}$$

Note that

$$\begin{aligned} u_t &= e^{-\alpha x - \beta t}(-\beta v + v_t) \\ u_x &= e^{-\alpha x - \beta t}(-\alpha v + v_x) \\ u_{xx} &= e^{-\alpha x - \beta t}\left(\alpha^2 v - 2\alpha v_x + v_{xx}\right) \end{aligned} \tag{6}$$

Substituting into the generalized heat equation (3) gives

$$v_t = v_{xx} + (b - 2\alpha) v_x + \left(\alpha^2 + \beta - \alpha b + c\right) v + g e^{\alpha x + \beta t} \tag{7}$$

To get rid of the v_x and v terms, we choose

$$\begin{aligned} b - 2\alpha &= 0 \\ \alpha^2 + \beta - \alpha b + c &= 0 \end{aligned}$$

Solving for α, β gives

$$\begin{aligned} \alpha &= b/2 \\ \beta &= -\left(\alpha^2 - \alpha b + c\right) = \frac{b^2}{4} - c \end{aligned}$$

Choosing

$$h(x,t) = g(x,t) e^{\alpha x + \beta t} = g(x,t) e^{\alpha x + \beta t}$$

gives the standard heat equation

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + h(x,t).$$

3 Problem 6

Consider the heat problem with periodic boundary conditions

$$\begin{aligned} u_t &= u_{xx} \\ u(0,t) &= 0; \qquad u(1,t) = \cos\omega t; \qquad t > 0 \\ u(x,0) &= f(x) \qquad 0 < x < 1. \end{aligned}$$

a. Prove that the steady-state solution, $u_{SS}(x,t)$, is unique.

Solution: The steady-state solution $u_{SS}(x,t)$ satisfies

$$\begin{aligned} (u_{SS})_t &= (u_{Ss})_{xx} \\ u_{SS}(0,t) &= 0; \qquad u_{SS}(1,t) = \cos\omega t; \qquad t > 0 \end{aligned}$$

$$u_{SS} \text{ periodic in time with period } \frac{2\pi}{\omega}$$

Consider two steady-state solutions u_1 and u_2. Let $v(x,t) = u_1 - u_2$. Then v satisfies

$$\begin{aligned} v_t &= v_{xx} \\ v(0,t) &= 0; \qquad v(1,t) = 0; \qquad t > 0 \end{aligned}$$

$$v(x,t) \text{ periodic in time with period } \frac{2\pi}{\omega}$$

From class, solutions to the heat equation with homogenous (zero) Type I BCs approach zero exponentially as $t \to \infty$,

$$\lim_{t\to\infty} v(x,t) = 0$$

But $v(x,t)$ is $2\pi/\omega$-periodic, and hence the only possibility is that $v(x,t) = 0$, which implies $u_1 = u_2$ and the steady-state $u_{SS}(x,t)$ is unique.

b. Find $u_{SS}(x,t)$ by using the complex change of variable $u_{SS}(x,t) = \text{Re}\left\{U(x)e^{i\omega t}\right\}$.

Solution: Following the steps in class. The solution is given by making the transformation $x \to 1 - x$ to our solution from class,

$$u_{SS}(x,t) = \text{Re}\left\{ \frac{\exp\left(\sqrt{\frac{\omega}{2}}(1+i)x\right) - \exp\left(-\sqrt{\frac{\omega}{2}}(1+i)x\right)}{\exp\left(\sqrt{\frac{\omega}{2}}(1+i)\right) - \exp\left(-\sqrt{\frac{\omega}{2}}(1+i)\right)} Ae^{i\omega t} \right\}.$$