Solutions to Problems for Quasi-Linear PDEs

18.303 Linear Partial Differential Equations

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1 Problem 1

Solve the traffic flow problem

$$\frac{\partial u}{\partial t} + (1-2u)\frac{\partial u}{\partial x} = 0, \qquad u(x,0) = f(x)$$

for an initial traffic group

$$f(x) = \begin{cases} \frac{1}{3}, & |x| > 1 \\ \frac{1}{2}\left(\frac{5}{3} - |x|\right), & |x| \le 1 \end{cases}$$

(a) At what time t_s and position x_s does a shock first form?

(b) Sketch the characteristics and indicate the region in the xt-plane in which the solution is well-defined (i.e. does not break down).

(c) Sketch the density profile $u = u(x,t)$ vs. x for several values of t in the interval $0 \le t \le t_s$.

Solution: (a) We can rewrite the PDE as

$$(1-2u, 1, 0) \cdot \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}, -1\right) = 0$$

We write t, x and u as functions of $(r;s)$, i.e. $t(r;s)$, $x(r;s)$, $u(r;s)$. We have written $(r;s)$ to indicate r is the variable that parametrizes the curve, while s is a parameter that indicates the position of the particular trajectory on the initial curve. Thus, the parametric solution is

$$\frac{dt}{dr} = 1, \qquad \frac{dx}{dr} = 1-2u, \qquad \frac{du}{dr} = 0$$

with initial condition on $r = 0$,

$$t(0;s) = 0, \qquad x(0;s) = s, \qquad u(0;s) = f(s).$$

where $s \in \mathbb{R}$. We find t and u first, since these can be found independently from one another. Integrating the ODEs and imposing the IC for t and u gives

$$t\,(r; s) = r, \qquad u\,(r; s) = f\,(s)\,. \tag{1}$$

Substituting for $u\,(r; s)$ into the ODE for $x\,(r; s)$ and integrating gives

$$x\,(r; s) = (1 - 2f\,(s))\,r + const$$

Imposing the IC $x\,(0; s) = s$ gives

$$x\,(r; s) = (1 - 2f\,(s))\,r + s. \tag{2}$$

Combining (1) and (2), the characteristics are

$$x = (1 - 2f\,(s))\,t + s = \begin{cases} \frac{1}{3}t + s, & |s| > 1 \\ \left(|s| - \frac{2}{3}\right)t + s, & |s| \le 1 \end{cases}$$

The first shock occurs at time

$$t_s = \frac{1}{2\max\{f'\,(s)\}} = \frac{1}{2\left(\frac{1}{2}\right)} = 1 \tag{3}$$

where the characteristics starting from $s = -1$ and $s = 0$ meet,

$$x_s = \frac{1}{3}t_s - 1 = -\frac{2}{3}t_s = -\frac{2}{3}.$$

(b) Figure 1 sketch shows the xt-plane up to the shock time $t = t_s$ and notes the important characteristics by thick solid lines. The thick characteristics divide the xt-plane into four regions. In R_1 and R_4, $|s| \ge 1$ and $u = f\,(s) = 1/3$. In R_2, $-1 \le s \le 0$, and for fixed t, u increases linearly in x from $1/3$ to $5/6$. In R_3, $0 \le s \le 1$ and u decreases linearly in x from $5/6$ to $1/3$.

(c) In Figure 2, we sketch the density profile $u = u\,(x, t)$ vs. x at times $t = 0$, $1/2$ and $t = t_s = 1$. To do so, we draw imaginary horizontal lines at $t = t_0$ in the xt-plot in part (b) and observe at what x-values these cross the important characteristics (thick black lines). We already know how u varies in each region, for fixed time. Thus once we know the x-values of the characteristics that start at $s = -1$, 0, 1, we draw the corresponding u-values $1/3$, $5/6$, $1/3$, and connect them with lines.

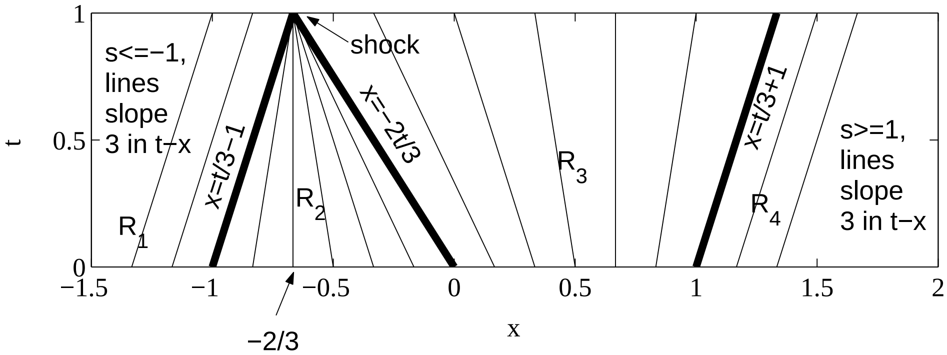

Figure 1: Sketch of characteristics up to the shock time $t = t_s = 1$. Thick lines are important characteristics.

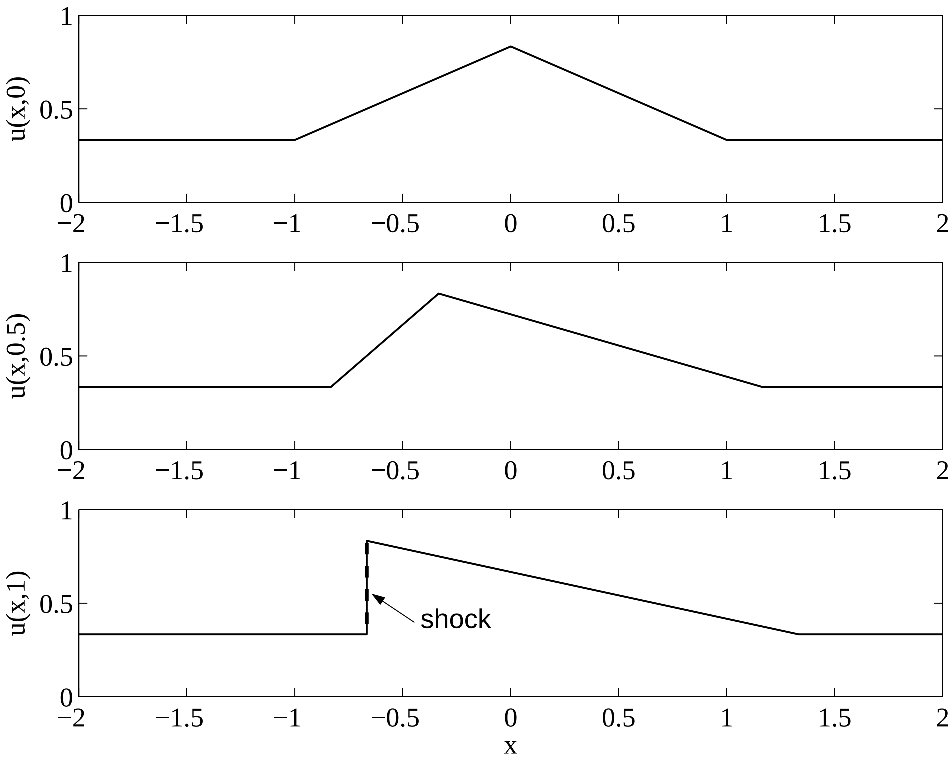

Figure 2: Sketch of density profiles $u = u(x, t)$ vs. x at times $t = 0$, $1/2$ and $t = t_s = 1$.

2 Problem 2 : Water waves

The surface displacement for shallow water waves is governed by (in scaled coordinates),

$$\left(1+\frac{3}{2}h\right)\frac{\partial h}{\partial x}+\frac{\partial h}{\partial t}=0$$

Here, $h=0$ is the mean free surface of the water. Consider the initial water wave profile

$$h\left(x,0\right)=f\left(x\right)=\begin{cases}\varepsilon\left(1+\cos x\right), & |x|\leq\pi\\ 0, & |x|>\pi\end{cases}\tag{4}$$

(a) Find the parametric solution and characteristic curves.

Solution: The parametric solution is given by

$$\frac{dt}{dr}=1,\qquad \frac{dh}{dr}=0,\qquad \frac{dx}{dr}=1+\frac{3}{2}h$$

with initial conditions $t\left(0\right)=0$, $x\left(0\right)=s$ and $h\left(x,0\right)=h\left(s,0\right)$. Solving the ODEs subject to the initial conditions gives the parametric solution

$$t=r,\qquad h=f\left(s\right),\qquad x=\left(1+\frac{3}{2}f\left(s\right)\right)t+s\tag{5}$$

for $s\in\mathbb{R}$.

(b) Show that two characteristics starting at $s=s_1$ and $s=s_2$ where $s_1,s_2\in(0,\pi)$ intersect at time

$$t_{int}=\frac{2}{3\varepsilon}\left(-\frac{s_1-s_2}{\cos s_1-\cos s_2}\right)$$

Show that

$$t_{int}\geq\frac{2}{3\varepsilon},\qquad \text{for all } s_1,s_2\in(0,\pi)$$

and

$$t_{int}\to\frac{2}{3\varepsilon},\qquad \text{as } s_1,s_2\to\frac{\pi}{2}$$

Thus the solution breaks down along the characteristics starting at $s=\pi/2$, when $t=t_c=2/\left(3\varepsilon\right)$.

Solution: From (5), the solutions starting at $s=s_1$ and $s=s_2$ where $s_1,s_2\in(0,\pi)$ (and, without loss of generality, $s_1<s_2$) intersect when

$$\left(1+\frac{3}{2}f\left(s_1\right)\right)t_{int}+s_1=x_{int}=\left(1+\frac{3}{2}f\left(s_2\right)\right)t_{int}+s_2$$

Solving for the time t_{int} gives

$$t_{int}=\frac{2}{3}\frac{s_2-s_1}{f\left(s_1\right)-f\left(s_2\right)}$$

Since $s_1, s_2 \in (0,\pi)$, substituting for $f(s)$ from (4) gives

$$\begin{aligned} t_{int} &= \frac{2}{3}\frac{s_2 - s_1}{\varepsilon(1+\cos s_1) - \varepsilon(1+\cos s_2)} \\ &= \frac{2}{3\varepsilon}\left(-\frac{s_1 - s_2}{\cos s_1 - \cos s_2}\right) \end{aligned} \tag{6}$$

By the mean value theorem,

$$\cos s_1 - \cos s_2 = -(s_1 - s_2)\sin\xi$$

for some $\xi \in [s_1, s_2] \subseteq (0,\pi)$, so that (6) becomes

$$t_{int} = \frac{2}{3\varepsilon}\frac{1}{\sin\xi} \tag{7}$$

For this range of $\xi \in [s_1, s_2] \subseteq (0,\pi)$, we have $0 < \sin\xi \leq 1$, so that (7) becomes

$$t_{int} = \frac{2}{3\varepsilon}\frac{1}{\sin\xi} \geq \frac{2}{3\varepsilon}$$

Note that as s_1, $s_2 \to \pi/2$, ξ also approaches $\pi/2$ and hence from (7),

$$\lim_{s_1, s_2 \to \pi/2} t_{int} = \lim_{\xi \to \pi/2} t_{int} = \frac{2}{3\varepsilon}$$

This implies that along the characteristic starting at $s = \pi/2$, the solution breaks down at $t = t_c = 2/(3\varepsilon)$. The x-value where the breakdown occurs is

$$x = \left(1 + \frac{3}{2}f\left(\frac{\pi}{2}\right)\right)\frac{2}{3\varepsilon} + \frac{\pi}{2} = \left(1 + \frac{3\varepsilon}{2}\cos\left(\frac{\pi}{2}\right)\right)\frac{2}{3\varepsilon} + \frac{\pi}{2} = \frac{2}{3\varepsilon} + \frac{\pi}{2}.$$

(c) Calculate $\partial h/\partial x$ using implicitly differentiation (the solution cannot be found explicitly) and hence show that along the characteristic starting at $s = \pi/2$,

$$\lim_{t \to t_c^-} \frac{\partial h}{\partial x} = -\infty$$

Thus the wave slope becomes vertical.

Solution: By the chain rule,

$$\frac{\partial h}{\partial x} = \frac{\partial h}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial h}{\partial s}\frac{\partial s}{\partial x} = 0 + f'(s)\frac{\partial s}{\partial x} = f'(s)\left(\frac{\partial x}{\partial s}\right)^{-1} = \frac{f'(s)}{\frac{3}{2}f'(s)\,t + 1} \tag{8}$$

Note that

$$f'(\pi/2) = -\varepsilon\sin\frac{\pi}{2} = -\varepsilon,$$

and hence

$$\frac{\partial h}{\partial x} = \frac{-\varepsilon}{-\frac{3}{2}\varepsilon t + 1}$$

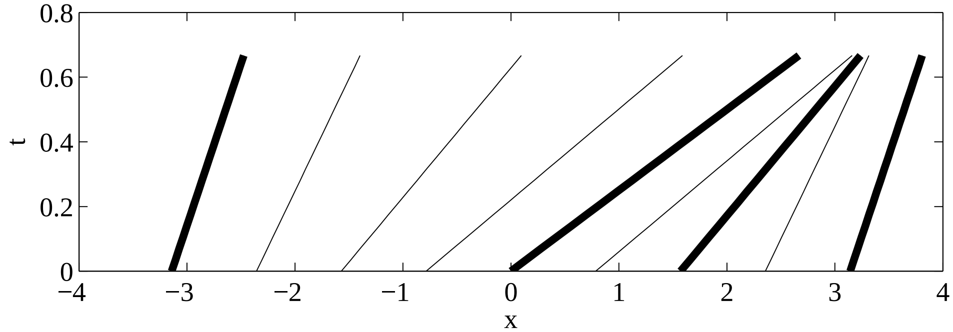

Figure 3: Sketch of characteristics up to the shock time $t = t_s = 2/3$. Thick lines are important characteristics. We took $\varepsilon = 1$.

Thus, the limit as $t \to t_c^-$ (where $t_c = 2/(3\varepsilon)$) is

$$\lim_{t \to t_c^-} \frac{\partial h}{\partial x} = \lim_{t \to t_c^-} \frac{-\varepsilon}{-\frac{3}{2}\varepsilon t + 1} = -\infty$$

(d) Sketch the wave profile $h(x, t_c)$, giving the x-values where the wave is vertical and where the maximum displacement occurs.

Note that the extrema of the displacement occurs where $\partial h/\partial x = 0$, or, from (8),

$$\frac{\partial h}{\partial x} = \frac{f'(s)}{\frac{3}{2}f'(s)\,t + 1} = 0 \qquad \Longleftrightarrow \qquad \varepsilon(-\sin x) = 0 \qquad \Longleftrightarrow \qquad x = 0, \pm\pi$$

I didn't ask for this, but to plot the wave profile, you need to know what the characteristics are doing. Figure 3 shows the important characteristics. Again, to find the wave profiles at a given time $t = t_0$, we draw an imaginary horizontal line at $t = t_0$ in the xt-plot of the characteristics and observe at what x-values this line cross the characteristics. We know the h values along each characteristic, and thus we can construct a table of x and corresponding h values at time $t = t_0$. Then we plot h vs. x. Figure 4 illustrates the wave profiles at $t = 0, 1/3, 2/3$, for $\varepsilon = 1$. The profile becomes vertical along the $s = \pi/2$ characteristic at time $t = 2/3$ at $x = 2/3 + \pi/2$. Come and see me if you have questions about how to do this - it's pretty simple once you get the hang of it.

The interpretation of the plot is that after a time $t = 2/3$ (recall $\varepsilon = 1$), the wave has moved a distance $x = 2/3$, it's tail has gotten longer, and it's front has steepened.

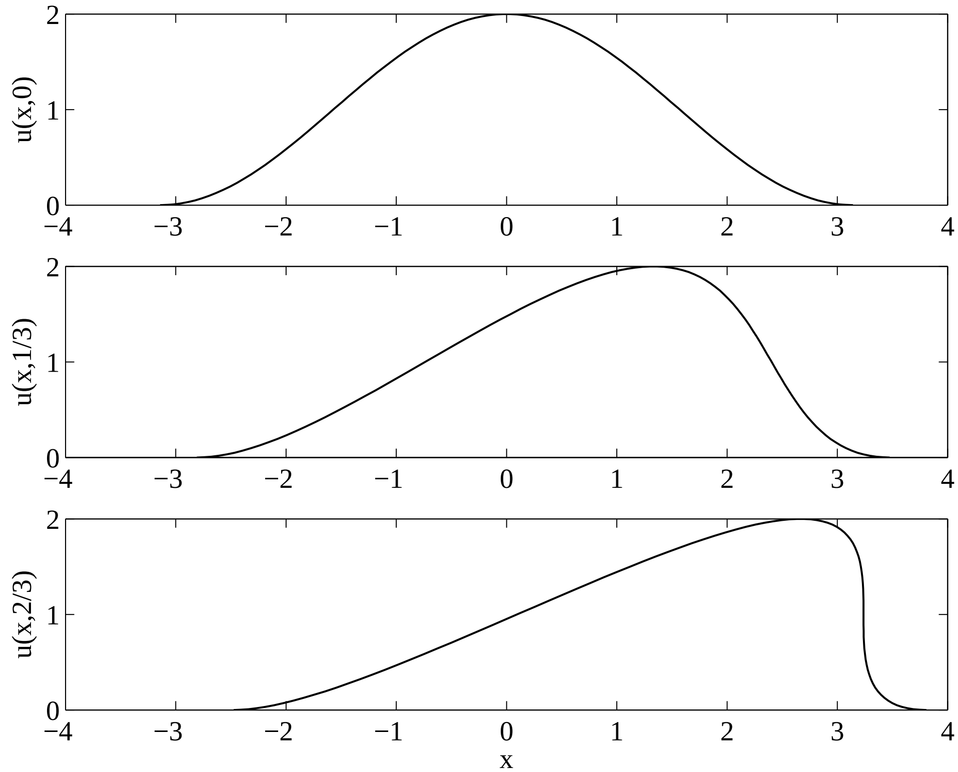

Figure 4: Sketch of wave profiles at times $t = 0,\ 1/3,\ 2/3$. At $t = 2/3$, the wave profile is vertical ($\partial h/\partial x = \infty$ at $x = 2/3 + \pi/2$, along the $s = \pi/2$ characteristic. Here, we took $\varepsilon = 1$.

3 Problem 3

Consider the quasi-linear PDE and initial condition

$$\begin{aligned} u_t + u\,u_x + \frac{1}{2}u &= 0, \qquad t > 0, \qquad -\infty < x < \infty \\ u(x,0) &= \varepsilon \sin x, \qquad -\infty < x < \infty \end{aligned}$$

where $\varepsilon > 0$ is constant.

(a) Find the parametric solution and characteristic curves.

Solution: The PDE can be written as

$$(A,B,C)\cdot(u_x,u_t,-1) = \left(u,1,-\frac{1}{2}u\right)\cdot(u_x,u_t,-1) = 0.$$

The characteristic curves are given by

$$\frac{dt}{dr} = B = 1, \qquad \frac{dx}{dr} = A = u, \qquad \frac{du}{dr} = C = -\frac{1}{2}u$$

The initial conditions at $r = 0$ are $t = 0$, $x = s$, $u = f(s) = \varepsilon \sin s$. Integrating the ODEs and imposing the ICs gives

$$t = r, \qquad u = f(s)\,e^{-r/2} = f(s)\,e^{-t/2}, \qquad x = 2f(s)\left(1 - e^{-r/2}\right) + s = 2f(s)\left(1 - e^{-t/2}\right) + s \tag{9}$$

where $f(s) = \varepsilon \sin s$.

(b) Give the solution u in implicit form by writing u in terms of x, t (but not r, s).

Solution: The second and third equations in (9) are

$$u = f(s)\,e^{-t/2}, \qquad x = 2f(s)\left(1 - e^{-t/2}\right) + s$$

Noting that $f(s) = \varepsilon \sin s = ue^{t/2}$, we have

$$\begin{aligned} x &= 2ue^{t/2}\left(1 - e^{-t/2}\right) + \arcsin\left(\frac{ue^{t/2}}{\varepsilon}\right) \\ &= 2u\left(e^{t/2} - 1\right) + \arcsin\left(\frac{ue^{t/2}}{\varepsilon}\right) \end{aligned}$$

Thus, the solution u is given implicitly via

$$\sin\left(x + 2u\left(1 - e^{t/2}\right)\right) = \frac{ue^{t/2}}{\varepsilon} \tag{10}$$

(c) For $\varepsilon = 1$, show that the solution first breaks down at $t = t_c = 2\ln 2$. Show that along the characteristic through $(x,t) = (\pi, 0)$, we have

$$\lim_{t\to t_c^-} u_x = -\infty.$$

Solution: The Jacobian is

$$\frac{\partial(x,t)}{\partial(r,s)} = \det\begin{pmatrix} x_r & x_s \\ t_r & t_s \end{pmatrix} = \det\begin{pmatrix} u & 2f'(s)\left(1-e^{-r/2}\right)+1 \\ 1 & 0 \end{pmatrix} = -2f'(s)\left(1-e^{-r/2}\right)-1$$

The solution breaks down when the Jacobian is zero, or

$$-2f'(s)\left(1-e^{-r/2}\right)-1=0$$

Since $r=t$ and $f'(s)=\varepsilon\cos s$, we have

$$2\varepsilon\cos s\left(1-e^{-t/2}\right)=-1 \tag{11}$$

Note that the breakdown must occur for $t>0$, since $t=0$ will not satisfy the above equation. Also, $\left(1-e^{-t/2}\right)>0$ since $t>0$. Thus the breakdown occurs when $\cos s<0$ and $t>0$. The smallest time for breakdown occurs at the most negative value of $\cos s$, i.e., $\cos s=-1$, when

$$1-\frac{1}{2\varepsilon}=e^{-t_c/2}$$

or

$$t_c=-2\ln\left(1-\frac{1}{2\varepsilon}\right)$$

Since $\varepsilon=1$, the first breakdown occurs at $t_c=2\ln 2$.

To find the s for the characteristic that passes through $(x,t)=(\pi,0)$, we substitute $t=0$, $x=\pi$ into the equation for x in (9),

$$\pi=x=2f(s)\left(1-e^{-t/2}\right)+s=s$$

Thus $s=\pi$. Substituting $s=\pi$ into (9) gives

$$\begin{aligned} x &= 2\varepsilon(\sin\pi)\left(1-e^{-t/2}\right)+\pi=\pi \\ u &= \varepsilon(\sin\pi)\,e^{-t/2}=0 \end{aligned}$$

Thus $x=\pi$ and $u=0$ along this characteristic. To find u_x, we differentiate (10) (with $\varepsilon=1$) implicitly with respect to x,

$$\cos\left(x+2u\left(1-e^{t/2}\right)\right)\left(1+2u_x\left(1-e^{t/2}\right)\right)=u_xe^{t/2}$$

Substituting $x=\pi$ and $u=0$ gives

$$-\left(1+2u_x\left(1-e^{t/2}\right)\right)=u_xe^{t/2}$$

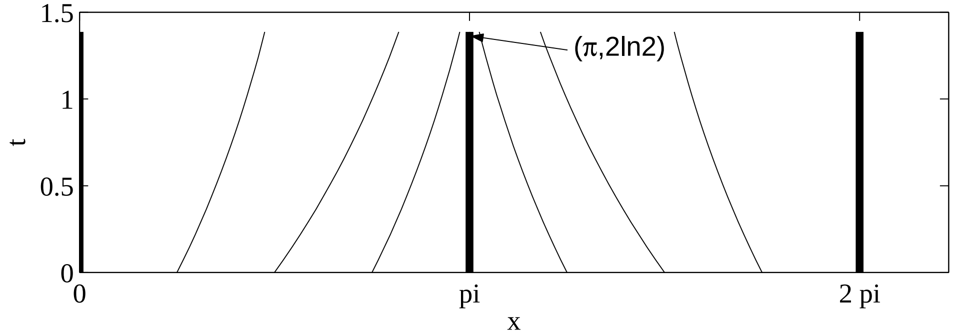

Figure 5: Sketch of characteristics up to the shock time $t = t_c = 2\ln 2$. Thick lines are important characteristics.

Solving for u_x gives

$$u_x = \frac{1}{e^{t/2} - 2}$$

For $s = \pi$, $\cos s = -1$, so that the solution breaks down along this characteristic at $t = t_c = 2\ln 2$. As $t \to t_c^-$, the limit of u_x is

$$\lim_{t \to t_c^-} u_x = \lim_{t \to t_c^-} \frac{1}{e^{t/2} - 2} = -\infty$$

(d) For $\varepsilon = 1$, sketch the characteristics and the solution profile at time t_c.

Solution: Since the initial condition is periodic, we must only plot the region $0 \le x \le 2\pi$, $t \ge 0$. The solution is repeated in the other regions $2(n-1)\pi \le x \le 2n\pi$, for all integers n. Note that $x = \pi$ is a line of symmetry. To see this, consider the characteristics $s = \pi/2$ and $s = 3\pi/2$ with $\varepsilon = 1$,

$$\begin{aligned} s &= \frac{\pi}{2} \quad \Longrightarrow \quad x = 2\left(1 - e^{-t/2}\right) + \frac{\pi}{2} \\ s &= \frac{\pi}{2} \quad \Longrightarrow \quad x = -2\left(1 - e^{-t/2}\right) + \frac{3\pi}{2} = -\left(2\left(1 - e^{-t/2}\right) + \frac{\pi}{2}\right) + 2\pi \end{aligned}$$

A few characteristics are plotted in Figure 5 up to the time $t = t_c$.

Substituting $\varepsilon = 1$ and $t = t_c = 2\ln 2$ into the implicit solution (10) gives

$$\sin(x - 2u) = 2u$$

and hence

$$x = 2u + \arcsin(2u)$$

Choosing values for u in $[0, 0.5]$, we compute the corresponding x-values. Just be careful that the angles arcsin returns can be in the first or second quadrant, so that you get two sets of x-values

$$\begin{aligned} x &= 2u + \arcsin(2u) \\ x &= 2u + \pi - \arcsin(2u) \end{aligned}$$

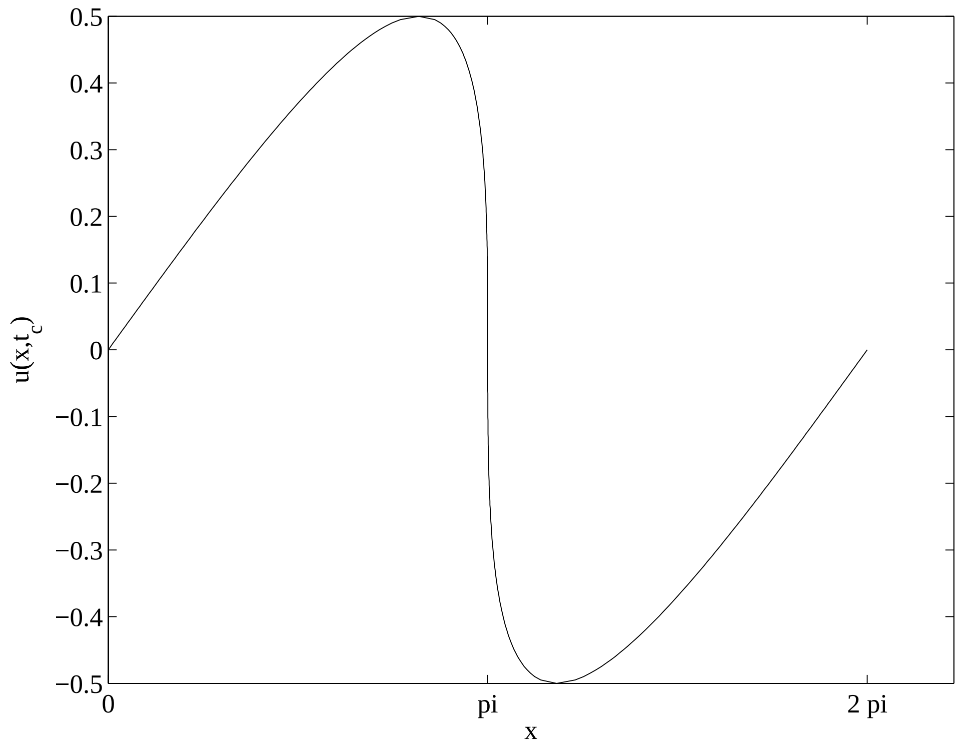

Figure 6: Sketch of $u(x, t_c)$ profile ($t_c = 2\ln 2$, $\varepsilon = 1$). Since $u(x,t)$ is 2π-periodic in x, the $u(x,t)$ is given by periodicity for values of x outside the region plotted.

Plotting these two sets of points gives you $u\,(x, t_c)$ in $[0, \pi]$. To get u in $[\pi, 2\pi]$, recall it is 2π periodic. We first find x for u in $[-0.5, 0]$ and then translate the resulting x-values by 2π. The plot is given in Figure 6.

(e) Show that the solution exists for all time if $0 < \varepsilon \leq 1/2$.

Solution: Recall that the solution breaks down if there is an s and t that satisfy Eq. (11),

$$2\varepsilon\,(\cos s)\left(1 - e^{-t/2}\right) = -1$$

For $0 < \varepsilon \leq 1/2$, we have $0 < 2\varepsilon \leq 1$ and for $t \geq 0$, $0 \leq 1 - e^{-t/2} < 1$, so that

$$\left|2\varepsilon\,(\cos s)\left(1 - e^{-t/2}\right)\right| < 1$$

Thus Eq. (11) cannot be satisfied, and the solution is valid for all time $t \geq 0$.