Problems for Infinite Spatial Domain Prolems and the Fourier Transform

18.303 Linear Partial Differential Equations

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1 Problem 1

(i) Show that

$$u(x,t) = u_0 \operatorname{erf}\left(\frac{x}{2\sqrt{t}}\right), \qquad t > 0, \qquad x \in \mathbb{R},$$

where u_0 is a constant, is a solution of the heat equation

$$u_t = u_{xx},$$

and satisfies the initial condition

$$u(x,0) = f(x) = \begin{cases} u_0, & \text{if } x > 0, \\ 0, & \text{if } x = 0, \\ -u_0, & \text{if } x < 0 \end{cases}$$

in the sense that

$$\lim_{t \to 0^+} u(x,t) = f(x).$$

NOTE: All that is required is a change of variable in the integral, and then writing the integral in terms of the error function erf. Also, $f(x)$ does not decay as $x \to \infty$, but it turns out this requirement can be relaxed as long as the integrals exist.

(ii) Give a physical interpretation of the solution. Sketch the curves $u(x,t) = const$ in the xt-plane.

(iii) Derive the solution (i) from the general solution we derived in class in terms of the heat kernel $K(s,x,t)$, using the initial temperature $u(x,0) = f(x)$.

2 Problem 2

(i) Find the temperature $u(x,t)$ of a semi-infinite rod $(x \geq 0)$, whose end $(x = 0)$ is kept at a temperature of zero, and with an initial hot-spot, $u(x,0) = f(x)$, where

$$f(x) = \begin{cases} u_0, & \text{if } x \in (x_0, x_1) \\ 0, & \text{if } x \in [0, x_0) \cup (x_1, \infty) \end{cases}$$

with x_0, x_1 constants, $0 \leq x_0 < x_1$. Sketch the temperature profiles $t = const$ (i.e., $u(x,t_0)$ in the ux-plane for various fixed times t_0), $x = const$ (i.e., $u(x_0,t)$ in the ut-plane for various fixed x_0) and the level curves $u(x,t) = const$ in the xt-plane. See note below.

(ii) Repeat (i) with the end of the rod $(x = 0)$ insulated. See note below.

(iii) Referring to (ii), show that the temperature of the insulated end is a maximum at time

$$t = \frac{x_1^2 - x_0^2}{4\kappa(\log x_1 - \log x_0)}$$

where κ is the thermal diffusivity.

NOTE: in both (i) and (ii), just use the general solution we derived in class with the heat kernel, by suitably extending $f(x)$ to the whole real line (i.e. odd extension or even extension - see class notes). The integrals in the solution can then be expressed as the sum of four terms involving error functions erf.

3 Problem 3

Show that

$$u(x,y) = \frac{2u_0}{\pi} \arctan\left(\frac{x}{y}\right)$$

where u_0 is constant, is a solution of Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

and satisfies the boundary condition

$$\lim_{y \to 0^+} u(x,y) = f(x)$$

Give a physical interpretation of the solution (i.e. how does this relate to what Heat Problem?). Sketch the isothermal curves (level curves) $u(x,y) = const$ in the xy-plane. Note that in polar coordinates,

$$\theta = \arctan\left(\frac{x}{y}\right)$$

where θ is the angle measured from the y-axis ($\theta = 0$ is the y-axis) and increasing counter-clockwise.