Final Examination

18.303 Linear Partial Differential Equations

Matthew J. Hancock

Dec. 21, 2006

Total points: 100

1 Rules [requires student signature!]

1. I will use only pencils, pens, erasers, and straight edges to complete this exam.
2. I will NOT use calculators, notes, books or other aides.

Signature: ____ Date: ____________.

Please hand in this question sheet with your solutions following the exam.

2 Note

Work on problems (and sub-parts) in any order; just be sure to label the question. Be sure to show a few key intermediate steps and make statements in words when deriving results - answers only will not get full marks. You are free to use any of the information given on the next two pages, without proof, on any question in the exam.

3 Given

You may use the following without proof:

The Laplacian ∇^2 in polar coordinates is

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

1D Sturm-Liouville Problems: The eigen-solution to

$$X'' + \lambda X = 0; \qquad X(0) = 0 = X(L)$$

is

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right), \qquad \lambda_n = \left(\frac{n\pi}{L}\right)^2, \qquad n = 1, 2, 3, ..$$

The eigen-solution to

$$Y'' + \lambda Y = 0; \qquad Y'(0) = 0 = Y'(L)$$

is

$$Y_n(x) = \cos\left(\frac{n\pi x}{L}\right), \qquad \lambda_n = \left(\frac{n\pi}{L}\right)^2, \qquad n = 0, 1, 2, 3, ..$$

Orthogonality condition, for any $L > 0$ (e.g. $L = 1$, π, $\pi/2$, etc)

$$\int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = \begin{cases} L/2, & m = n, \\ 0, & m \neq n. \end{cases}$$

$$\int_0^L \sin\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)dx = 0$$

The general solution to Bessel's Equation

$$r\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \left(\lambda r^2 - m^2\right)R(r) = 0, \qquad m = 0, 1, 2, 3, ...$$

is

$$R_m(r) = c_{m1}J_m\left(\sqrt{\lambda}r\right) + c_{m2}Y_m\left(\sqrt{\lambda}r\right)$$

where c_{mn} are constants of integration, $J_m\left(\sqrt{\lambda}r\right)$ is bounded as $r \to 0$ and

$$\left|Y_m\left(\sqrt{\lambda}r\right)\right| \to \infty \text{ as } r \to 0.$$

Orthogonality for Bessel Functions J_n,

$$\int_0^1 rJ_n(j_{n,m}r)\,J_k(j_{k,l}r)\,dr = 0, \qquad \text{if } n \neq k \text{ or } m \neq l$$

where $j_{n,m}$ is the m'th zero of the Bessel function of order n. If $n = k$ and $m = l$, just write

$$\int_0^1 r\left(J_n\left(j_{n,m}r\right)\right)^2 dr > 0$$

A result derived from the Divergence Theorem,

$$\int\int_D v\nabla^2 v dV = -\int\int_D |\nabla v|^2 dV + \int_{\partial D} v\nabla v \cdot \hat{\mathbf{n}} dS \tag{1}$$

for any 2D or 3D region D with closed boundary ∂D.

The Jacobian determinant of the change of variable $(r,s) \to (x,t)$ is

$$\frac{\partial(x,t)}{\partial(r,s)} = \det\begin{pmatrix} x_r & x_s \\ t_r & t_s \end{pmatrix} = x_r t_s - x_s t_r = \frac{\partial x}{\partial r}\frac{\partial t}{\partial s} - \frac{\partial x}{\partial s}\frac{\partial t}{\partial r}$$

Rayleigh Quotient:

$$R(v) = \frac{\int\int\int_D \nabla v \cdot \nabla v dV}{\int\int\int_D v^2 dV} = \frac{\int\int\int_D |\nabla v|^2 dV}{\int\int\int_D v^2 dV}$$

Trig identities:

$$\begin{aligned}
\sin a \sin b &= \frac{1}{2}\left(\cos(a-b) - \cos(a+b)\right) \\
\cos a \cos b &= \frac{1}{2}\left(\cos(a-b) + \cos(a+b)\right) \\
\sin(a+b) &= \sin a \cos b + \sin b \cos a \\
\cos(a+b) &= \cos a \cos b - \sin a \sin b
\end{aligned}$$

The spatial Fourier Transform of $u(x,t)$ and $f(x)$ are defined as

$$\begin{aligned}
\overline{U}(\omega,t) &= \mathcal{F}[u(x,t)](\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x,t)e^{i\omega x}dx \\
F(\omega) &= \mathcal{F}[f(x)](\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{i\omega x}dx
\end{aligned}$$

The Inverse Fourier Transforms of $\overline{U}(\omega,t)$ and $F(\omega)$ are defined as

$$\begin{aligned}
u(x,t) &= \mathcal{F}^{-1}\left[\overline{U}(\omega,t)\right](x) = \int_{-\infty}^{\infty} \overline{U}(\omega,t)e^{-i\omega x}d\omega \\
f(x) &= \mathcal{F}^{-1}[F(\omega)](x) = \int_{-\infty}^{\infty} F(\omega)e^{-i\omega x}d\omega
\end{aligned}$$

The IFT of a Gaussian is

$$\mathcal{F}^{-1}\left[e^{-\alpha\omega^2}\right] = \sqrt{\frac{\pi}{\alpha}}e^{-x^2/4\alpha} \tag{2}$$

where α can involve constants or variables, but must be independent of ω and x.