Solutions to Problems for Infinite Spatial Domain Prolems and the Fourier Transform

18.303 Linear Partial Differential Equations

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1 Problem 1

(i) Show that

$$u(x,t) = u_0 \mathrm{erf}\left(\frac{x}{2\sqrt{t}}\right), \qquad t > 0, \qquad x \in \mathbb{R}, \tag{1}$$

where u_0 is a constant, is a solution of the heat equation

$$u_t = u_{xx}, \tag{2}$$

and satisfies the initial condition

$$u(x,0) = f(x) = \begin{cases} u_0, & \text{if } x > 0, \\ 0, & \text{if } x = 0, \\ -u_0, & \text{if } x < 0 \end{cases} \tag{3}$$

in the sense that

$$\lim_{t \to 0^+} u(x,t) = f(x).$$

Solution: We show by direct substitution that $u(x,t)$ is the solution to the heat equation and initial condition. First, note that the error function is defined by

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-s^2} ds \tag{4}$$

Thus by the chain rule,

$$\begin{aligned}
u_t &= u_0 \frac{2}{\sqrt{\pi}} \exp\left(-\frac{x^2}{4t}\right) \frac{x}{2t^{3/2}} \left(-\frac{1}{2}\right) = -\frac{u_0 x}{2\sqrt{\pi} t^{3/2}} \exp\left(-\frac{x^2}{4t}\right) \\
u_{xx} &= \frac{\partial^2}{\partial x^2} \left(u_0 \mathrm{erf}\left(\frac{x}{2\sqrt{t}}\right)\right) = \frac{\partial}{\partial x} \left(u_0 \frac{2}{\sqrt{\pi}} \frac{1}{2\sqrt{t}} \exp\left(-\frac{x^2}{4t}\right)\right) \\
&= u_0 \frac{2}{\sqrt{\pi}} \exp\left(-\frac{x^2}{4t}\right) \frac{1}{2\sqrt{t}} \left(-\frac{2x}{4t}\right) \\
&= -\frac{u_0 x}{2\sqrt{\pi} t^{3/2}} \exp\left(-\frac{x^2}{4t}\right) \\
&= u_t
\end{aligned}$$

Thus Eq. (1) for $u(x,t)$ satisfies the Heat Equation (2).

We now show that (1) satisfies the IC (3). Note that

$$\begin{aligned}
x &> 0 : \lim_{t\to 0^+} u(x,t) = u_0 \mathrm{erf}\left(\lim_{t\to 0^+} \frac{x}{2\sqrt{t}}\right) = u_0 \mathrm{erf}(\infty) = u_0 \\
x &= 0 : \lim_{t\to 0^+} u(0,t) = \lim_{t\to 0^+} 0 = 0 \\
x &< 0 : \lim_{t\to 0^+} u(x,t) = u_0 \mathrm{erf}\left(\lim_{t\to 0^+} \frac{x}{2\sqrt{t}}\right) = u_0 \mathrm{erf}(-\infty) = -u_0
\end{aligned}$$

Thus, $\lim_{t\to 0^+} u(x,t) = f(x)$.

(ii) Give a physical interpretation of the solution. Sketch the curves $u(x,t) = \mathit{const}$ in the xt-plane.

Solution: $u(x,t)$ is the temperature in an infinite rod with thermal diffusivity 1 and initial temperature u_0 for $x > 0$ and $-u_0$ for $x < 0$. You could also think of this as two infinite rods put together, with the temperature at $x = 0$ held at 0. The level curves are given by

$$u(x,t) = u_0 \mathrm{erf}\left(\frac{x}{2\sqrt{t}}\right) = \mathit{const}$$

and hence

$$\frac{x}{2\sqrt{t}} = \mathit{const} \quad \Longrightarrow \quad t = cx^2$$

These are parabolas pointing upward, and are plotted in Figure 1.

(iii) Derive the solution (i) from the general solution we derived in class in terms of the heat kernel $K(s,x,t)$, using the initial temperature $u(x,0) = f(x)$. NOTE: All that is required is a change of variable in the integral, and then writing the integral in terms of the error function erf. Also, $f(x)$ does not decay as $x \to \infty$, but it turns out this requirement can be relaxed as long as the integrals exist.

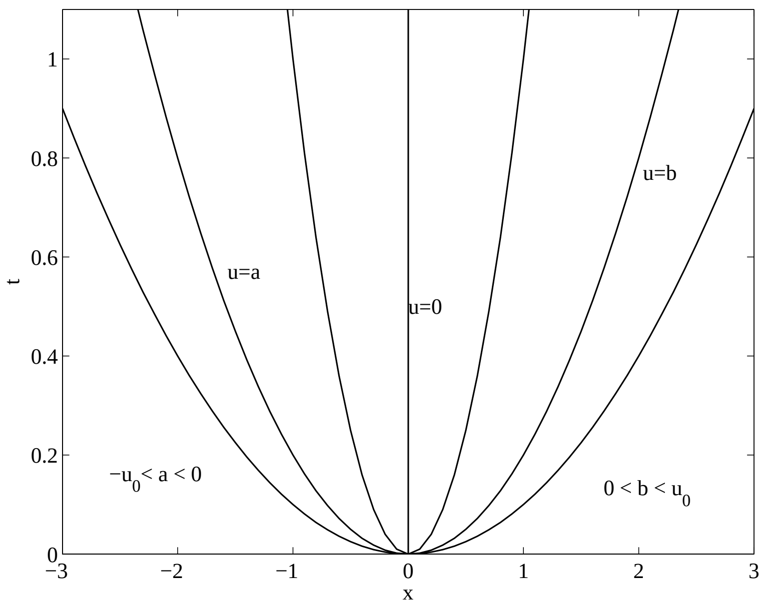

Figure 1: Level curves of $u(x,t)$ for question 1(ii).

Solution: In class, we derived the solution of the heat equation with general initial condition $f(x)$,

$$u(x,t) = \int_{-\infty}^{\infty} \frac{f(s)}{\sqrt{4\pi\kappa t}} \exp\left(-\frac{(x-s)^2}{4\kappa t}\right) ds$$

Substituting $f(s)$ from (3) gives

$$u(x,t) = -\frac{u_0}{\sqrt{4\pi\kappa t}} \int_{-\infty}^{0} \exp\left(-\frac{(x-s)^2}{4\kappa t}\right) ds + \frac{u_0}{\sqrt{4\pi\kappa t}} \int_{0}^{\infty} \exp\left(-\frac{(x-s)^2}{4\kappa t}\right) ds$$

Making the change of variable

$$\alpha = \frac{s-x}{\sqrt{4\kappa t}}$$

in the integrals gives

$$\begin{aligned}
u(x,t) &= -\frac{u_0}{\pi^{1/2}} \int_{-\infty}^{-x/\sqrt{4\kappa t}} e^{-\alpha^2} d\alpha + \frac{u_0}{\pi^{1/2}} \int_{-x/\sqrt{4\kappa t}}^{\infty} e^{-\alpha^2} ds \\
&= -\frac{u_0}{\pi^{1/2}} \left(\int_{-\infty}^{0} e^{-\alpha^2} d\alpha + \int_{0}^{-x/\sqrt{4\kappa t}} e^{-\alpha^2} d\alpha \right) \\
&\quad + \frac{u_0}{\pi^{1/2}} \left(\int_{-x/\sqrt{4\kappa t}}^{0} e^{-\alpha^2} ds + \int_{0}^{\infty} e^{-\alpha^2} ds \right) \\
&= \frac{2u_0}{\pi^{1/2}} \int_{-x/\sqrt{4\kappa t}}^{0} e^{-\alpha^2} ds \\
&= u_0 \frac{2}{\pi^{1/2}} \int_{0}^{x/\sqrt{4\kappa t}} e^{-\alpha^2} ds \qquad (5)
\end{aligned}$$

since

$$\int_{-\infty}^{0} e^{-\alpha^2} d\alpha = \int_{0}^{\infty} e^{-\alpha^2} ds.$$

Substituting the definition of the error function (4) into (5) and setting $\kappa = 1$ gives

$$u(x,t) = u_0 \text{erf}\left(\frac{x}{2\sqrt{t}}\right)$$

as required.

2 Problem 2

(i) Find the temperature $u(x,t)$ of a semi-infinite rod ($x \geq 0$), whose end ($x = 0$) is kept at a temperature of zero, and with an initial hot-spot, $u(x,0) = f(x)$, where

$$f(x) = \begin{cases} u_0, & \text{if } x \in (x_0, x_1) \\ 0, & \text{if } x \in [0, x_0) \cup (x_1, \infty) \end{cases} \tag{6}$$

with x_0, x_1 constants, $0 \leq x_0 < x_1$. Sketch the temperature profiles $t = const$ (i.e., $u(x,t_0)$ in the ux-plane for various fixed times t_0), $x = const$ (i.e., $u(x_0,t)$ in the ut-plane for various fixed x_0) and the level curves $u(x,t) = const$ in the xt-plane. See note below.

(ii) Repeat (i) with the end of the rod ($x = 0$) insulated. See note below.

(iii) Referring to (ii), show that the temperature of the insulated end is a maximum at time

$$t = \frac{x_1^2 - x_0^2}{4\kappa(\log x_1 - \log x_0)}$$

where κ is the thermal diffusivity.

NOTE: in both (i) and (ii), just use the general solution we derived in class with the heat kernel, by suitably extending $f(x)$ to the whole real line (i.e. odd extension or even extension - see class notes). The integrals in the solution can then be expressed as the sum of four terms involving error functions erf.

Solution (i): In class, we derived that solution to the heat equation on a semi-infinite rod ($x \geq 0$) whose end is kept at zero is

$$u(x,t) = \int_{-\infty}^{\infty} \frac{\tilde{f}(s)}{\sqrt{4\pi\kappa t}} \exp\left(-\frac{(x-s)^2}{4\kappa t}\right) ds \tag{7}$$

where $\tilde{f}(s)$ is the odd extension of $f(s)$,

$$\tilde{f}(s) = \begin{cases} f(s), & s > 0 \\ 0, & s = 0 \\ -f(-s), & s < 0 \end{cases} \tag{8}$$

Substituting (8) into (7) gives

$$u(x,t) = -\int_{-\infty}^{0} \frac{f(-s)}{\sqrt{4\pi\kappa t}} \exp\left(-\frac{(x-s)^2}{4\kappa t}\right) ds + \int_{0}^{\infty} \frac{f(s)}{\sqrt{4\pi\kappa t}} \exp\left(-\frac{(x-s)^2}{4\kappa t}\right) ds$$

Substituting for $f(s)$ from (6) gives

$$u(x,t) = -\frac{u_0}{\sqrt{4\pi\kappa t}} \int_{-x_1}^{-x_0} \exp\left(-\frac{(x-s)^2}{4\kappa t}\right) ds + \frac{u_0}{\sqrt{4\pi\kappa t}} \int_{x_0}^{x_1} \exp\left(-\frac{(x-s)^2}{4\kappa t}\right) ds$$

Making the change of variable

$$\alpha = \frac{s-x}{\sqrt{4\kappa t}}$$

gives

$$\begin{aligned} u(x,t) &= -\frac{u_0}{\sqrt{\pi}} \int_{\frac{-x_1-x}{\sqrt{4\kappa t}}}^{\frac{-x_0-x}{\sqrt{4\kappa t}}} e^{-\alpha^2} d\alpha + \frac{u_0}{\sqrt{\pi}} \int_{\frac{x_0-x}{\sqrt{4\kappa t}}}^{\frac{x_1-x}{\sqrt{4\kappa t}}} e^{-\alpha^2} d\alpha \\ &= \frac{u_0}{\sqrt{\pi}} \left(-\int_{\frac{-x_1-x}{\sqrt{4\kappa t}}}^{0} e^{-\alpha^2} d\alpha - \int_{0}^{\frac{-x_0-x}{\sqrt{4\kappa t}}} e^{-\alpha^2} d\alpha + \int_{\frac{x_0-x}{\sqrt{4\kappa t}}}^{0} e^{-\alpha^2} d\alpha + \int_{0}^{\frac{x_1-x}{\sqrt{4\kappa t}}} e^{-\alpha^2} d\alpha \right) \\ &= \frac{u_0}{\sqrt{\pi}} \left(-\int_{0}^{\frac{x_1+x}{\sqrt{4\kappa t}}} e^{-\alpha^2} d\alpha + \int_{0}^{\frac{x_0+x}{\sqrt{4\kappa t}}} e^{-\alpha^2} d\alpha - \int_{0}^{\frac{x_0-x}{\sqrt{4\kappa t}}} e^{-\alpha^2} d\alpha + \int_{0}^{\frac{x_1-x}{\sqrt{4\kappa t}}} e^{-\alpha^2} d\alpha \right) \end{aligned}$$

Substituting the definition of the error function (4) gives

$$u(x,t) = \frac{u_0}{2} \left(-\mathrm{erf}\left(\frac{x_1+x}{\sqrt{4\kappa t}}\right) + \mathrm{erf}\left(\frac{x_0+x}{\sqrt{4\kappa t}}\right) - \mathrm{erf}\left(\frac{x_0-x}{\sqrt{4\kappa t}}\right) + \mathrm{erf}\left(\frac{x_1-x}{\sqrt{4\kappa t}}\right) \right)$$

The temperature profiles and the level curves are plotted in Figures 2 to 5.

Solution (ii): In class, we derived that solution to the heat equation on a semi-infinite rod ($x \geq 0$) whose end is insulated,

$$u(x,t) = \int_{-\infty}^{\infty} \frac{\tilde{f}(s)}{\sqrt{4\pi\kappa t}} \exp\left(-\frac{(x-s)^2}{4\kappa t}\right) ds \tag{9}$$

where $\tilde{f}(s)$ is the odd extension of $f(s)$,

$$\tilde{f}(s) = \begin{cases} f(s), & s > 0 \\ f(-s), & s < 0 \end{cases} \tag{10}$$

Substituting (10) into (9) gives

$$u(x,t) = \int_{-\infty}^{0} \frac{f(-s)}{\sqrt{4\pi\kappa t}} \exp\left(-\frac{(x-s)^2}{4\kappa t}\right) ds + \int_{0}^{\infty} \frac{f(s)}{\sqrt{4\pi\kappa t}} \exp\left(-\frac{(x-s)^2}{4\kappa t}\right) ds$$

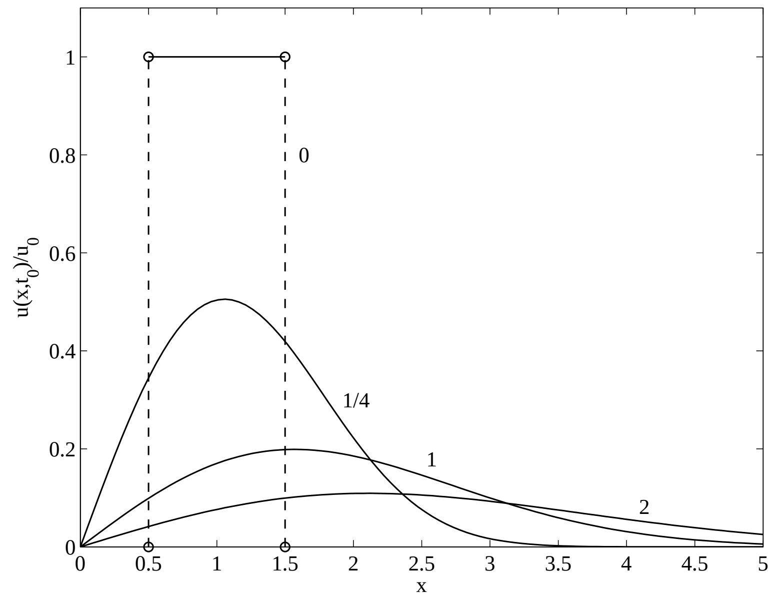

Figure 2: $u(x,t_0)$ curves for question 2(i). Numbers adjacent to curves indicate time t_0.

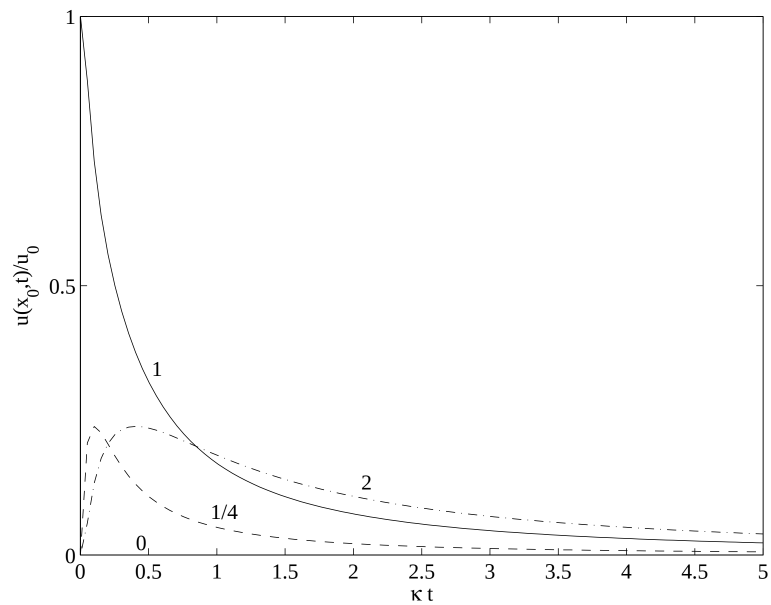

Figure 3: $u(x_0,t)$ curves for question 2(i). Numbers adjacent to curves indicate position x_0.

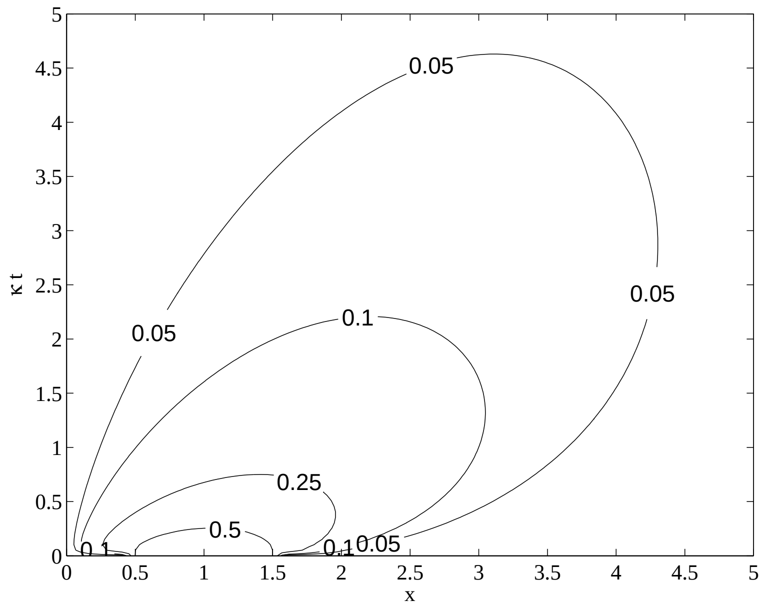

Figure 4: Level curves of $u(x,t)$ for question 2(i). Numbers adjacent to curves indicate value of u on level curve.

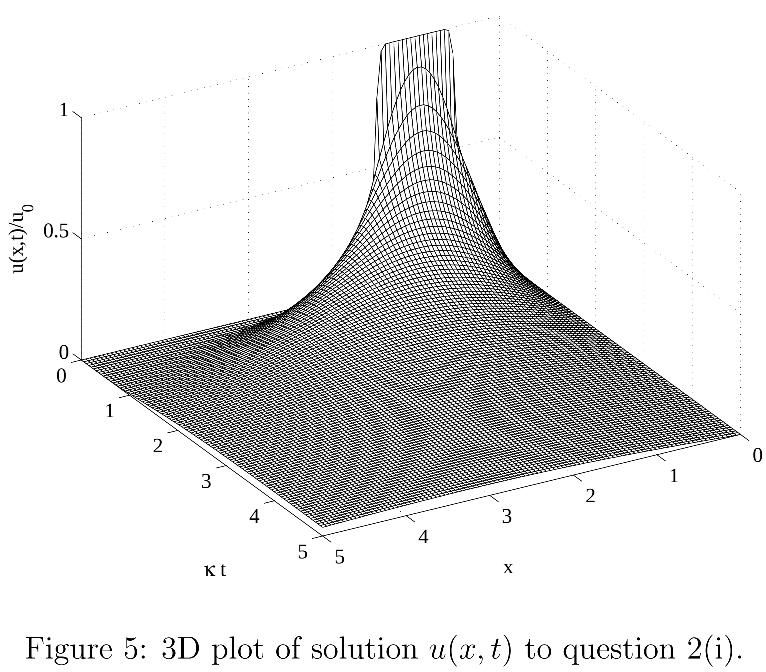

Figure 5: 3D plot of solution $u(x,t)$ to question 2(i).

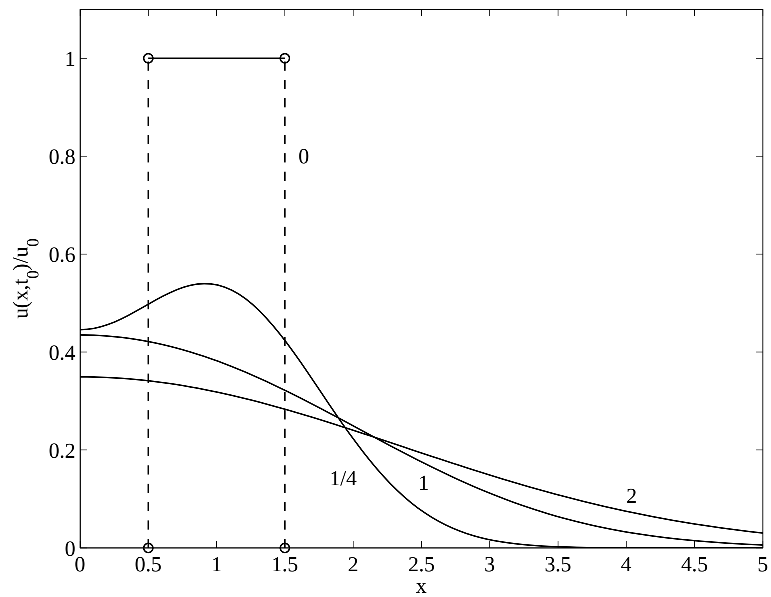

Figure 6: $u(x, t_0)$ curves for question 2(ii). Numbers adjacent to curves indicate time t_0.

Substituting for $f(s)$ from (6) gives

$$u(x,t) = \frac{u_0}{\sqrt{4\pi\kappa t}} \int_{-x_1}^{-x_0} \exp\left(-\frac{(x-s)^2}{4\kappa t}\right) ds + \frac{u_0}{\sqrt{4\pi\kappa t}} \int_{x_0}^{x_1} \exp\left(-\frac{(x-s)^2}{4\kappa t}\right) ds$$

Making the change of variable

$$\alpha = \frac{s-x}{\sqrt{4\kappa t}}$$

gives

$$\begin{aligned}
u(x,t) &= \frac{u_0}{\sqrt{\pi}} \int_{\frac{-x_1-x}{\sqrt{4\kappa t}}}^{\frac{-x_0-x}{\sqrt{4\kappa t}}} e^{-\alpha^2} d\alpha + \frac{u_0}{\sqrt{\pi}} \int_{\frac{x_0-x}{\sqrt{4\kappa t}}}^{\frac{x_1-x}{\sqrt{4\kappa t}}} e^{-\alpha^2} d\alpha \\
&= \frac{u_0}{\sqrt{\pi}} \left(\int_{\frac{-x_1-x}{\sqrt{4\kappa t}}}^{0} e^{-\alpha^2} d\alpha + \int_{0}^{\frac{-x_0-x}{\sqrt{4\kappa t}}} e^{-\alpha^2} d\alpha + \int_{\frac{x_0-x}{\sqrt{4\kappa t}}}^{0} e^{-\alpha^2} d\alpha + \int_{0}^{\frac{x_1-x}{\sqrt{4\kappa t}}} e^{-\alpha^2} d\alpha \right) \\
&= \frac{u_0}{\sqrt{\pi}} \left(\int_{0}^{\frac{x_1+x}{\sqrt{4\kappa t}}} e^{-\alpha^2} d\alpha - \int_{0}^{\frac{x_0+x}{\sqrt{4\kappa t}}} e^{-\alpha^2} d\alpha - \int_{0}^{\frac{x_0-x}{\sqrt{4\kappa t}}} e^{-\alpha^2} d\alpha + \int_{0}^{\frac{x_1-x}{\sqrt{4\kappa t}}} e^{-\alpha^2} d\alpha \right)
\end{aligned}$$

Substituting the definition of the error function (4) gives

$$u(x,t) = \frac{u_0}{2} \left(\operatorname{erf}\left(\frac{x_1+x}{\sqrt{4\kappa t}}\right) - \operatorname{erf}\left(\frac{x_0+x}{\sqrt{4\kappa t}}\right) - \operatorname{erf}\left(\frac{x_0-x}{\sqrt{4\kappa t}}\right) + \operatorname{erf}\left(\frac{x_1-x}{\sqrt{4\kappa t}}\right) \right)$$

The temperature profiles and the level curves are plotted in Figures 6 to 9.

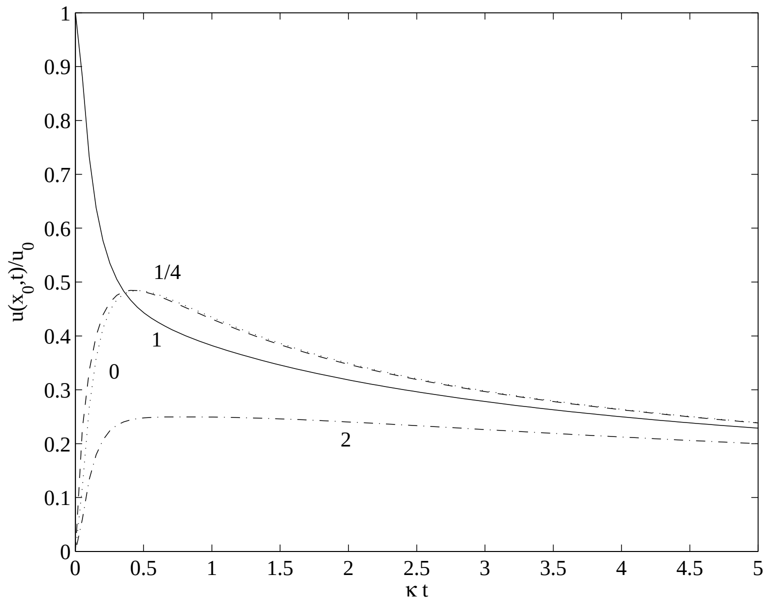

Figure 7: $u(x_0, t)$ curves for question 2(ii). Numbers adjacent to curves indicate position x_0.

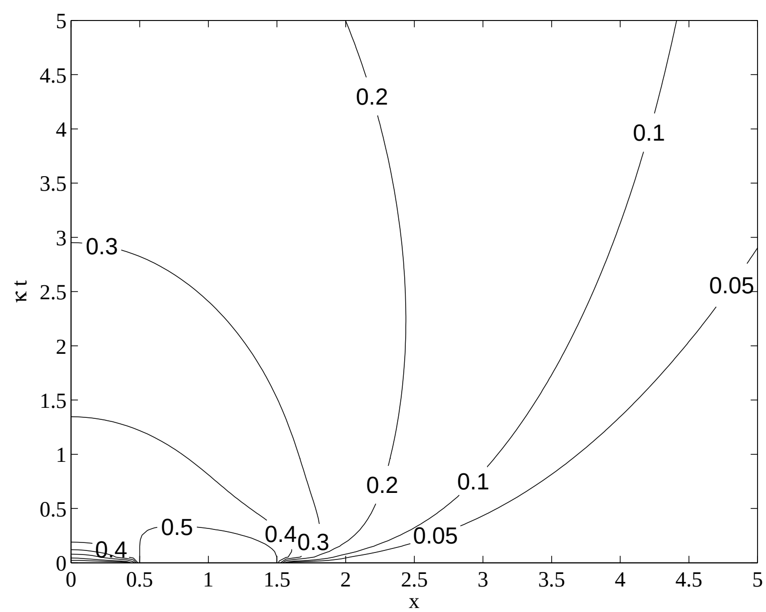

Figure 8: Level curves of $u(x, t)$ for question 2(ii). Numbers adjacent to curves indicate value of u on level curve.

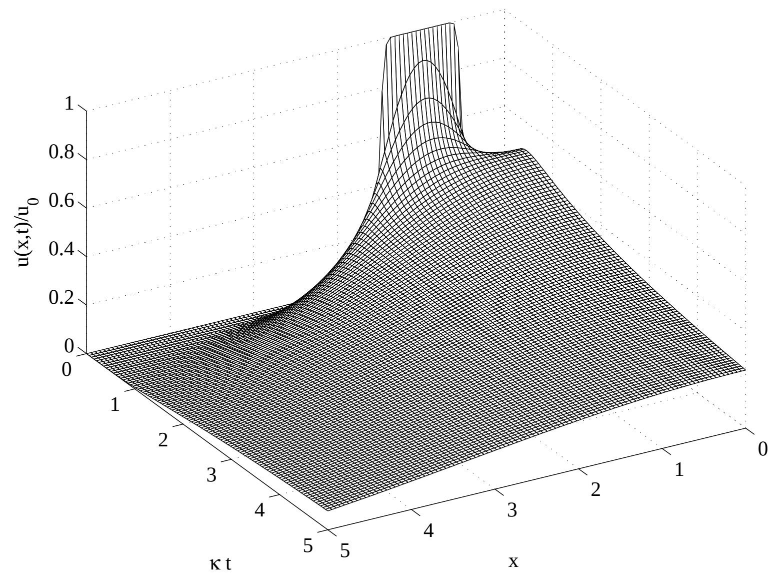

Figure 9: 3D plot of solution $u(x,t)$ to question 2(ii).

Solution (iii): The temperature at the insulated end of the solution in problem (ii) is

$$u(0,t) = u_0 \left(\operatorname{erf}\left(\frac{x_1}{\sqrt{4\kappa t}} \right) - \operatorname{erf}\left(\frac{x_0}{\sqrt{4\kappa t}} \right) \right)$$

The temperature is maximum when $u_t(0,t) = 0$,

$$0 = u_t(0,t) = -\frac{u_0}{4\sqrt{\kappa} t^{3/2}} \left(x_1 \exp\left(-\frac{x_1^2}{4\kappa t} \right) - x_0 \exp\left(-\frac{x_0^2}{4\kappa t} \right) \right)$$

Rearranging gives

$$x_1 = x_0 \exp\left(\frac{x_1^2 - x_0^2}{4\kappa t} \right)$$

Taking the log of both sides and solving for t yields

$$t = \frac{x_1^2 - x_0^2}{4\kappa \log(x_1/x_0)} = \frac{x_1^2 - x_0^2}{4\kappa (\log(x_1) - \log(x_0))}$$

3 Problem 3

Show that

$$u(x,y) = \frac{2u_0}{\pi} \arctan\left(\frac{x}{y} \right) \tag{11}$$

where u_0 is constant, is a solution of Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

and satisfies the boundary condition

$$\lim_{y\to 0^+} u(x,y) = f(x)$$

Give a physical interpretation of the solution (i.e. how does this relate to what Heat Problem?). Sketch the isothermal curves (level curves) $u(x,y) = const$ in the xy-plane. Note that in polar coordinates,

$$\theta = \arctan\left(\frac{x}{y}\right)$$

where θ is the angle measured from the y-axis ($\theta = 0$ is the y-axis) and increasing clockwise.

Solution: We can show $u(x,y)$ is a solution of Laplace's equation directly,

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{2u_0}{\pi}\frac{1}{1+(x/y)^2}\frac{1}{y}, \qquad \frac{\partial^2 u}{\partial x^2} = -\frac{2u_0}{\pi}\frac{2xy}{(y^2+x^2)^2} \\
\frac{\partial u}{\partial y} &= \frac{2u_0}{\pi}\frac{1}{1+(x/y)^2}\frac{-x}{y^2}, \qquad \frac{\partial^2 u}{\partial y^2} = \frac{2u_0}{\pi}\frac{2xy}{(y^2+x^2)^2}
\end{aligned}$$

Thus

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -\frac{2u_0}{\pi}\frac{2xy}{(y^2+x^2)^2} + \frac{2u_0}{\pi}\frac{2xy}{(y^2+x^2)^2} = 0$$

and hence (11) is a solution of Laplace's equation. The limit $y \to 0^+$ of (11) is

$$\begin{aligned}
x &> 0: \lim_{y\to 0^+} u(x,y) = \frac{2u_0}{\pi}\lim_{y\to 0^+}\arctan\left(\frac{x}{y}\right) = \frac{2u_0}{\pi}\arctan(\infty) = \frac{2u_0}{\pi}\frac{\pi}{2} = u_0 \\
x &= 0: \lim_{y\to 0^+} u(0,y) = \frac{2u_0}{\pi}\lim_{y\to 0^+} 0 = 0 \\
x &< 0: \lim_{y\to 0^+} u(x,y) = \frac{2u_0}{\pi}\lim_{y\to 0^+}\arctan\left(\frac{x}{y}\right) = \frac{2u_0}{\pi}\arctan(-\infty) = -\frac{2u_0}{\pi}\frac{\pi}{2} = -u_0
\end{aligned}$$

Thus

$$\lim_{y\to 0^+} u(x,y) = f(x)$$

where $f(x)$ is given in (3) in problem 1.

We can also show that (11) is a solution of Laplace's equation by substituting $f(s)$ into the general solution we derived in class using the Fourier Transform,

$$\begin{aligned}
u(x,y) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(s)\frac{2y}{(x-s)^2+y^2}ds \\
&= \frac{1}{2\pi}\left(\int_{-\infty}^{0} f(s)\frac{2y}{(x-s)^2+y^2}ds + \int_{0}^{\infty} f(s)\frac{2y}{(x-s)^2+y^2}ds\right) \\
&= \frac{yu_0}{\pi}\left(-\int_{-\infty}^{0}\frac{ds}{(x-s)^2+y^2} + \int_{0}^{\infty}\frac{ds}{(x-s)^2+y^2}\right)
\end{aligned}$$

Note that

$$\begin{aligned}
\int_a^b \frac{ds}{(s-x)^2+y^2} &= \int_{a-x}^{b-x} \frac{d\alpha}{\alpha^2+y^2} = \left[\frac{1}{y}\arctan\left(\frac{\alpha}{y}\right)\right]_{\alpha=a-x}^{b-x} \\
&= \frac{1}{y}\arctan\left(\frac{b-x}{y}\right) - \frac{1}{y}\arctan\left(\frac{a-x}{y}\right)
\end{aligned}$$

Thus

$$\begin{aligned}
u(x,y) &= \frac{yu_0}{\pi}\left(-\frac{1}{y}\arctan\left(\frac{-x}{y}\right) + \frac{1}{y}\arctan\left(\frac{-\infty-x}{y}\right)\right) \\
&\quad + \frac{yu_0}{\pi}\left(\frac{1}{y}\arctan\left(\frac{\infty-x}{y}\right) - \frac{1}{y}\arctan\left(\frac{-x}{y}\right)\right) \\
&= \frac{u_0}{\pi}\left(2\arctan\left(\frac{x}{y}\right) - \frac{\pi}{2} + \frac{\pi}{2}\right) \\
&= \frac{2u_0}{\pi}\arctan\left(\frac{x}{y}\right)
\end{aligned}$$

as required.

Physical interpretation: the solution $u(x,y)$ is the steady-state temperature of the upper half plane with boundary condition $u(x,0) = f(x)$. Since $\theta = \arctan(x/y)$ where θ is the angle measured from the y-axis ($\theta = 0$ is the y-axis) and increasing clockwise, we have

$$u(x,y) = u_0 \frac{\theta}{\pi/2}$$

The level curves $u = \mathit{const}$ are thus lines through the origin, $\theta = \mathit{const}$, and are sketched in Figure 10.

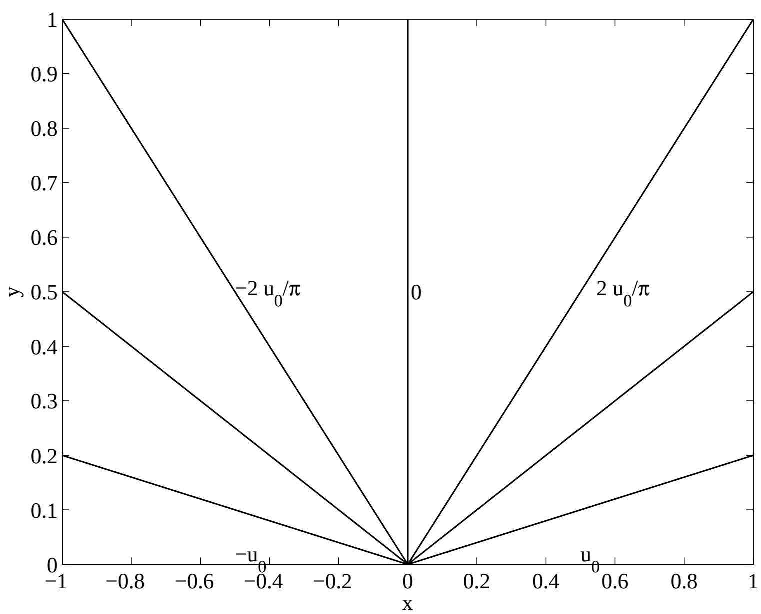

Figure 10: Level curves of $u(x, y)$ for question 3.