Solutions to Problems for the 1-D Wave Equation

18.303 Linear Partial Differential Equations

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1 Problem 1

(i) Generalize the derivation of the wave equation where the string is subject to a damping force $-b\partial u/\partial t$ per unit length to obtain

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} - 2k\frac{\partial u}{\partial t} \tag{1}$$

All variables will be left in dimensional form in this problem to make things a little different. How is the constant k related to b? What are the dimensions of b and k? The constant 2 is included for later convenience.

Solution: The derivation follows that in Section 1 of WaveEqnI.pdf. Consider an element of the string between x and $x + \Delta x$. Let $T(x,t)$ be tension and $\theta(x,t)$ be the angle wrt the horizontal x-axis. Note that

$$\tan\theta(x,t) = \text{slope of tangent at } (x,t) \text{ in } ux\text{-plane} = \frac{\partial u}{\partial x}(x,t). \tag{2}$$

Newton's Second Law ($F = ma$) states that

$$F = (\rho\Delta x)\frac{\partial^2 u}{\partial t^2} \tag{3}$$

where ρ is the linear density of the string (ML^{-1}) and Δx is the length of the segment. The force F comes from the tension in the spring and also the damping force (we ignore any external forces such as gravity). The damping force acts in the opposite direction to the motion. Recall our assumptions on the string. We assumed the displacements of the string are sufficiently small so that each point on the string moves vertically. Thus the damping force also acts vertically in the opposite direction to the motion. Balancing the forces in the horizontal direction gives

$$T(x+\Delta x,t)\cos\theta(x+\Delta x,t) = T(x,t)\cos\theta(x,t) = \tau = const \tag{4}$$

where τ is the constant horizontal tension. Balancing the forces in the vertical direction yields

$$\begin{aligned} F &= T(x+\Delta x,t)\sin\theta(x+\Delta x,t) - T(x,t)\sin\theta(x,t) - b\frac{\partial u}{\partial t}\Delta x \\ &= T(x+\Delta x,t)\cos\theta(x+\Delta x,t)\tan\theta(x+\Delta x,t) - T(x,t)\cos\theta(x,t)\tan\theta(x,t) - b\frac{\partial u}{\partial t}\Delta x \end{aligned} \tag{5}$$

Substituting (4) and (2) into (5) yields

$$\begin{aligned} F &= \tau\left(\tan\theta(x+\Delta x,t) - \tan\theta(x,t)\right) - b\frac{\partial u}{\partial t}\Delta x \\ &= \tau\left(\frac{\partial u}{\partial x}(x+\Delta x,t) - \frac{\partial u}{\partial x}(x,t)\right) - b\frac{\partial u}{\partial t}\Delta x. \end{aligned} \tag{6}$$

Substituting F from (3) into Eq. (6) and dividing by Δx gives

$$\rho\frac{\partial^2 u}{\partial t^2}(\xi,t) = \tau\frac{\frac{\partial u}{\partial x}(x+\Delta x,t) - \frac{\partial u}{\partial x}(x,t)}{\Delta x} - b\frac{\partial u}{\partial t}$$

for $\xi \in [x, x+\Delta x]$. Dividing by ρ and letting $\Delta x \to 0$ gives the 1-D Wave Equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} - 2k\frac{\partial u}{\partial t}, \qquad c^2 = \frac{\tau}{\rho} > 0, \qquad k = \frac{b}{2\rho}. \tag{7}$$

Note that c has units $[c] = \left[\frac{\text{Force}}{\text{Density}}\right]^{1/2} = LT^{-1}$ of speed, b has units $[b] = \left[\frac{\text{force}}{\text{distance} \times \text{speed}}\right] = \frac{MLT^{-2}}{L^2T^{-1}} = ML^{-1}T^{-1}$ and k has units $[k] = [b]/[\rho] = \frac{ML^{-1}T^{-1}}{ML^{-1}} = T^{-1}$. Thus k is proportional to a frequency (i.e. has units of 1/time or Hz).

(ii) Use separation of variables to find the normal modes of the damped Wave Equation (1) subject to the BCs

$$u(0,t) = 0 = u(l,t) \tag{8}$$

Impose a restriction on the parameters c, l, k which will guarantee that all solutions are oscillatory in time. You may assume that the eigenvalues and eigenfunctions are

$$\lambda_n = \frac{n^2\pi^2}{l^2}, \qquad X_n(x) = \sin\frac{n\pi x}{l}, \qquad n = 1, 2, 3...$$

Solution: Separating variables as

$$u(x,t) = X(x)T(t),$$

substituting into the PDE (1) and dividing by $c^2 X(x) T(t)$ gives

$$\frac{1}{c^2}\frac{T''}{T} + \frac{2k}{c^2}\frac{T'}{T} = \frac{X''}{X} = -\lambda \tag{9}$$

From the BCs (8),

$$X(0) = X(l) = 0$$

since $T(t)$ cannot be zero for all time to obtain a non-trivial solution. The Sturm-Liouville problem for $X(x)$ is, from (9),

$$X'' + \lambda X = 0; \qquad X(0) = X(l) = 0.$$

The non-trivial solutions are the eigenvalues and eigenfunctions

$$\lambda_n = \frac{n^2\pi^2}{l^2}, \qquad X_n(x) = \sin\frac{n\pi x}{l}. \tag{10}$$

The solutions for $T(t)$ are, from (9),

$$T'' + 2kT' + \lambda c^2 T = 0.$$

From (9), the values of λ for $T(t)$ are the same as those for $X(x)$. From (10), the only values of λ that lead to non-trivial solutions are $\lambda = \lambda_n$. Thus for each λ_n, we need to solve for the corresponding $T(t) = T_n(t)$ to find the normal mode $u_n(x,t) = X_n(x) T_n(t)$. To find the solutions for $T(t)$, we substitute $T = e^{rt}$,

$$r^2 + 2kr + \lambda c^2 = 0.$$

Solving the quadratic equation for r gives

$$r = -k \pm \sqrt{k^2 - c^2\lambda} \tag{11}$$

The solutions $T_n(t)$ corresponding to the eigenvalues $\lambda = \lambda_n$ are all oscillatory if r has a complex part, i.e. if $k^2 - c^2\lambda_n < 0$ for all n, or

$$k^2 < \min_{n\geq 1} c^2\lambda_n = \min_{n\geq 1}\frac{c^2 n^2\pi^2}{l^2} = \frac{c^2\pi^2}{l^2}.$$

Taking the square root of both sides and rearranging gives the criterion that all the $T_n(t)$ be oscillatory,

$$\frac{c\pi}{kl} > 1. \tag{12}$$

The solution $T_n(t)$ is a linear combination of e^{rt} where r is given in (11) with $\lambda = \lambda_n$,

$$T_n(t) = c_1 e^{\left(-k+\sqrt{k^2-c^2\lambda_n}\right)t} + c_2 e^{\left(-k-\sqrt{k^2-c^2\lambda_n}\right)t}$$

Under condition (12), $\sqrt{k^2 - c^2\lambda_n} = i\sqrt{c^2\lambda_n - k^2}$, where $c^2\lambda_n - k^2 > 0$. Thus

$$\begin{aligned} T_n(t) &= e^{-kt}\left(c_{n1}e^{i\sqrt{c^2\lambda_n-k^2}t} + c_{n2}e^{-i\sqrt{c^2\lambda_n-k^2}t}\right) \\ &= e^{-kt}\left((c_{n1}+c_{n2})\cos\left(\sqrt{c^2\lambda_n-k^2}t\right) + i(c_{n1}-c_{n2})\sin\left(\sqrt{c^2\lambda_n-k^2}t\right)\right) \\ &= e^{-kt}\left(\alpha_n\cos\left(\sqrt{c^2\lambda_n-k^2}t\right) + \beta_n\sin\left(\sqrt{c^2\lambda_n-k^2}t\right)\right) \end{aligned}$$

where we have rewritten the constants of integration as $\alpha_n = c_{n1} + c_{n2}$, $\beta_n = i(c_{n1} - c_{n2})$.

The corresponding normal modes are

$$u_n(x,t) = X_n(x)T_n(t) = e^{-kt}\left(\alpha_n\cos\left(\sqrt{c^2\lambda_n-k^2}t\right) + \beta_n\sin\left(\sqrt{c^2\lambda_n-k^2}t\right)\right)\sin\left(\frac{n\pi x}{l}\right) \tag{13}$$

where $\lambda_n = n^2\pi^2/l^2$. Notice that with damping ($k > 0$), the normal mode decays with time and oscillates ($c\pi > kl$) as it decays.

(iii) Express the frequency $\widetilde{f}_n$ of the oscillatory part of the n'th normal mode in terms of the frequency of the undamped mode $f_n = nc/(2l)$. What difference does the damping make?

Solution: The frequency $\widetilde{f}_n$ of the normal mode $u_n(x,t)$, given in (13), is

$$\widetilde{f}_n = \frac{1}{2\pi}\sqrt{c^2\lambda_n - k^2} = \frac{1}{2\pi}\sqrt{\frac{c^2n^2\pi^2}{l^2} - k^2} = \frac{cn}{2l}\sqrt{1 - \left(\frac{lk}{cn\pi}\right)^2} = f_n\sqrt{1 - \left(\frac{lk}{cn\pi}\right)^2} \tag{14}$$

As the damping ($k > 0$) increases, the frequencies of the normal modes decrease.

(iv) Show that the solution of the damped wave equation (1) subject to the BCs (8) and the initial condition

$$u(x,0) = f(x), \qquad \frac{\partial u}{\partial t}(x,0) = 0 \tag{15}$$

is given by

$$u(x,t) = e^{-kt}\sum_{n=1}^{\infty}\left(\alpha_n\cos\left(2\pi\widetilde{f}_n t\right) + \beta_n\sin\left(2\pi\widetilde{f}_n t\right)\right)\sin\left(\frac{n\pi x}{l}\right)$$

Express the constants α_n, β_n in terms of the Fourier Sine coefficients B_n of f.

Solution: Summing the normal modes gives the solution

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = e^{-kt}\sum_{n=1}^{\infty}\left(\alpha_n\cos\left(2\pi\widetilde{f}_n t\right) + \beta_n\sin\left(2\pi\widetilde{f}_n t\right)\right)\sin\left(\frac{n\pi x}{l}\right) \tag{16}$$

where the frequencies $\widetilde{f}_n$ are given in (14) and α_n, β_n are to be determined from the initial conditions on $u(x,t)$. Imposing the ICs (15) gives

$$\begin{aligned} f(x) &= u(x,0) = \sum_{n=1}^{\infty} \alpha_n \sin\left(\frac{n\pi x}{l}\right) && (17) \\ 0 &= u_t(x,0) = \sum_{n=1}^{\infty} \left(-k\alpha_n + 2\pi \widetilde{f}_n \beta_n\right) \sin\left(\frac{n\pi x}{l}\right) && (18) \end{aligned}$$

Eq. (18) implies

$$-k\alpha_n + 2\pi \widetilde{f}_n \beta_n = 0 \tag{19}$$

for all n. Multiplying both sides of (17) by $\sin m\pi x$, integrating from $x = 0$ to l, and using the orthogonality properties of $\sin m\pi x$, gives

$$\alpha_n = \frac{2}{l} \int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx. \tag{20}$$

From (19),

$$\beta_n = \frac{k\alpha_n}{2\pi \widetilde{f}_n} \tag{21}$$

From (16), (20) and (21), the complete solution is

$$u(x,t) = e^{-kt} \sum_{n=1}^{\infty} \alpha_n \left(\cos\left(2\pi \widetilde{f}_n t\right) + \frac{k}{2\pi \widetilde{f}_n} \sin\left(2\pi \widetilde{f}_n t\right)\right) \sin\left(\frac{n\pi x}{l}\right) \tag{22}$$

2 Problem 2

Prove that if a vibrating string is damped, i.e. subject to the PDE in Problem 1(i), then the energy $E(t)$ is monotone decreasing. You may use the formula we derived in lecture,

$$E(t) = \frac{\rho}{2} \int_0^l \left(u_t^2 + c^2 u_x^2\right) dx \tag{23}$$

Also, you may assume Homogeneous Type I BCs for the displacement $u(x,t)$.

Solution: The formula derived in lecture is valid for a system with damping, since the kinetic and potential energies of the string only depend on the displacement $u(x,t)$ and its derivatives. Differentiating (23) gives

$$\frac{dE}{dt} = \rho \int_0^l \left(u_t u_{tt} + c^2 u_x u_{xt}\right) dx$$

Replacing u_{tt} using the PDE (1) gives

$$\begin{aligned} \frac{dE}{dt} &= \rho \int_0^l \left(c^2 u_t u_{xx} - 2k u_t^2 + c^2 u_x u_{xt}\right) dx \\ &= \rho \int_0^l \left(c^2 \left(u_t u_x\right)_x - 2k u_t^2\right) dx \\ &= \rho c^2 \left[u_t u_x\right]_{x=0}^l - 2k\rho \int_0^l u_t^2 dx \end{aligned} \tag{24}$$

Differentiating the BCs in time t gives

$$\frac{d}{dt} u\left(0, t\right) = 0, \qquad \frac{d}{dt} u\left(l, t\right) = 0$$

Therefore, (24) becomes

$$\frac{dE}{dt} = -2k\rho \int_0^l u_t^2 dx \tag{25}$$

Thus $dE/dt \le 0$ which shows that the energy $E\left(t\right)$ is monotone non-increasing.

We want to show little more, namely that $E\left(t\right)$ is monotone decreasing for $t > 0$. We must simply show that there is no time interval $0 < t_1 \le t \le t_2$ in which $dE/dt = 0$. Suppose, for the sake of contradiction, there was a time interval $0 < t_1 \le t \le t_2$ in which $dE/dt = 0$. Then (25) implies u_t^2 must vanish identically for all $0 \le x \le l$ and for each time in this interval,

$$u_t\left(x, t\right) = 0, \qquad t \in \left[t_1, t_2\right], \qquad x \in \left[0, l\right]$$

Differentiating in time gives

$$u_{tt}\left(x, t\right) = 0, \qquad t \in \left[t_1, t_2\right], \qquad x \in \left[0, l\right]$$

Substituting for u_{tt} from the Damped Wave Equation (1), we have

$$u_{xx}\left(x, t\right) = 0, \qquad t \in \left[t_1, t_2\right], \qquad x \in \left[0, l\right]$$

Integrating in x and applying the BCs (8) gives

$$u\left(x, t\right) = 0, \qquad t \in \left[t_1, t_2\right], \qquad x \in \left[0, l\right]$$

Substituting the solution (22) from Problem 1 yields

$$e^{-kt} \sum_{n=1}^{\infty} \alpha_n \left(\cos\left(2\pi \widetilde{f}_n t\right) + \frac{k}{2\pi \widetilde{f}_n} \sin\left(2\pi \widetilde{f}_n t\right)\right) \sin\left(\frac{n\pi x}{l}\right), \qquad t \in \left[t_1, t_2\right], \qquad x \in \left[0, l\right]$$

From the orthogonality of $\sin(n\pi x/l)$, this implies that for all n,

$$e^{-kt}\alpha_n \left(\cos\left(2\pi \widetilde{f}_n t\right) + \frac{k}{2\pi \widetilde{f}_n} \sin\left(2\pi \widetilde{f}_n t\right)\right) = 0, \qquad t \in [t_1, t_2] \tag{26}$$

We assume that $f(x) \neq 0$, for otherwise the solution is trivial $u(x,t) = 0$. Thus from Eq. (20), there exists an n_1 such that

$$\alpha_{n_1} = \frac{2}{l} \int_0^l f(x) \sin\left(\frac{n_1 \pi x}{l}\right) dx \neq 0.$$

For this n_1, $\alpha_{n_1} \neq 0$ and (26) implies

$$\cos\left(2\pi \widetilde{f}_n t\right) + \frac{k}{2\pi \widetilde{f}_n} \sin\left(2\pi \widetilde{f}_n t\right) = 0$$

Rearranging yields

$$\tan\left(2\pi \widetilde{f}_n t\right) = -\frac{2\pi \widetilde{f}_n}{k}, \qquad t \in [t_1, t_2]$$

This equation makes no sense because the l.h.s. varies with time $t \in [t_1, t_2]$, while the r.h.s. is constant. This is a contradiction, and hence there can be no time interval where $dE/dt = 0$, unless the solution $u(x,t)$ is trivial, i.e. identically zero.

Therefore, for non-trivial solutions $u(x,t)$ (given by functions $f(x)$ that are not identically zero), the energy is monotonic decreasing, $dE/dt < 0$ for all time $t > 0$.

3 Problem 3

(i) Suppose that an "infinite string" has an initial displacement

$$u(x,0) = f(x) = \begin{cases} x+1, & -1 \leq x \leq 0 \\ -x+1, & 0 \leq x \leq 1 \\ 0, & |x| > 1 \end{cases}$$

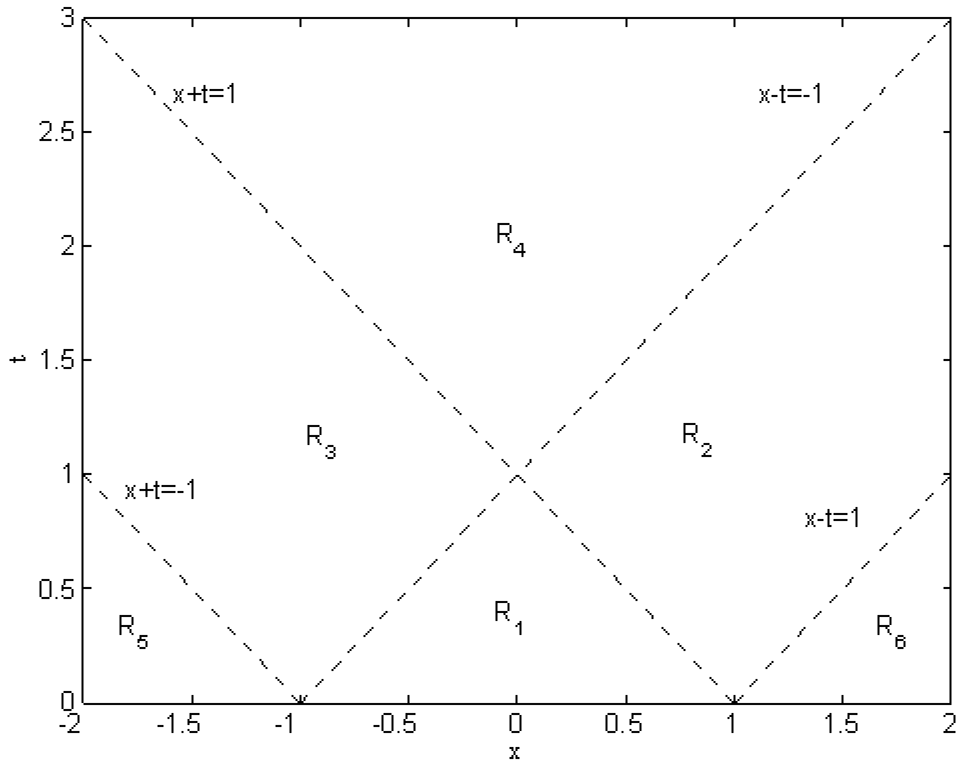
and zero initial velocity $u_t(x,0) = 0$. Write down the solution of the wave equation

$$u_{tt} = u_{xx}$$

with ICs $u(x,0) = f(x)$ and $u_t(x,0) = 0$ using D'Alembert's formula. Illustrate the nature of the solution by sketching the ux-profiles $z = u(x,t)$ of the string for $t = 0, 1/2, 1, 3/2$.

Solution: Step 1. Since $u_t(x,0) = 0$, D'Alembert's solution is

$$u(x,t) = \frac{f(x-t) + f(x+t)}{2}. \tag{27}$$



Notice that df/dx is not continuous, but it turns out that the smoothness criterion can be relaxed. $f(x)$ is piecewise smooth. Lastly, we can rewrite $f(x)$ as

$$f(x) = \begin{cases} 1 - |x|, & |x| \leq 1 \\ 0, & |x| > 1 \end{cases} \tag{28}$$

Step 2. Identify the regions. The function $f(x)$ is a triangle with vertices at $x = -1, 0, 1$. It is zero for $|x| > 1$. Thus, the regions of interest are found by plotting the four characteristics $x \pm t = \pm 1$ (see plot above). The regions are identified in the plot, and are given mathematically by

$$\begin{aligned} R_1 &= \{(x,t) : -1 \leq x - t \leq 1 \quad \text{and} \quad -1 \leq x + t \leq 1\} \\ R_2 &= \{(x,t) : -1 \leq x - t \leq 1 \quad \text{and} \quad x + t \geq 1\} \\ R_3 &= \{(x,t) : x - t \leq -1 \quad \text{and} \quad -1 \leq x + t \leq 1\} \\ R_4 &= \{(x,t) : x - t \leq -1 \quad \text{and} \quad x + t \geq 1\} \\ R_5 &= \{(x,t) : x + t \leq -1\}, \\ R_6 &= \{(x,t) : x - t \geq 1\} \end{aligned} \tag{29}$$

The regions determine where $x-t$ and $x+t$ are relative to ± 1, which tells us what part of the case function $f(x)$ should be used.

Step 3. Consider the solution in each region. In R_1, combining the inequalities gives $|x \pm t| \leq 1$ and hence from (28),

$$\begin{aligned} f(x-t) &= 1-|x-t| \\ f(x+t) &= 1-|x+t| \end{aligned}$$

thus (27) becomes

$$u(x,t) = \frac{f(x-t)+f(x+t)}{2} = 1 - \frac{1}{2}\left(|x-t|+|x+t|\right)$$

In region R_2,

$$\begin{aligned} f(x+t) &= 0 \\ f(x-t) &= 1-|x-t| \end{aligned}$$

and hence (27) becomes

$$u(x,t) = \frac{f(x-t)}{2} = \frac{1}{2} - \frac{1}{2}|x-t|$$

In region R_3,

$$\begin{aligned} f(x+t) &= 1-|x+t| \\ f(x-t) &= 0 \end{aligned}$$

and hence (27) becomes

$$u(x,t) = \frac{f(x+t)}{2} = \frac{1}{2} - \frac{1}{2}|x+t|$$

In regions R_4, R_5 and R_6, $f(x+t) = 0 = f(x-t)$ and hence $u = 0$. To summarize,

$$u(x,t) = \begin{cases} 1 - \frac{1}{2}\left(|x-t|+|x+t|\right), & (x,t) \in R_1 \\ \frac{1}{2} - \frac{1}{2}|x-t|, & (x,t) \in R_2 \\ \frac{1}{2} - \frac{1}{2}|x+t|, & (x,t) \in R_3 \\ 0, & (x,t) \in R_4, R_5, R_6 \end{cases} \tag{30}$$

Step 4. For each specific time $t = t_0$, write the x-intervals corresponding to the sets R_n (i.e. the intersection of the set R_n with $\{t = t_0\}$, or in the figure above, where the line $t = t_0$

intersects the region R_n). As a check, we note that at $t=0$,

$$\begin{aligned}
R_1 \cap \{t=0\} &= \{-1 \le x \le 1\} = \{|x| \le 1\} \\
R_2 \cap \{t=0\} &= \{x=1\} \\
R_3 \cap \{t=0\} &= \{x=-1\} \\
R_4 \cap \{t=0\} &= \varnothing \\
R_5 \cap \{t=0\} &= \{x \le -1\}, \\
R_6 \cap \{t=0\} &= \{x \ge 1\}
\end{aligned} \tag{31}$$

and (30) becomes

$$u(x,0) = \begin{cases} 1-|x|, & |x| \le 1 \\ 0, & |x| > 1 \end{cases} = f(x)$$

At $t = 1/2$,

$$\begin{aligned}
R_1 \cap \left\{t=\frac{1}{2}\right\} &= \left\{-\frac{1}{2} \le x \le \frac{3}{2} \quad \text{and} \quad -\frac{3}{2} \le x \le \frac{1}{2}\right\} = \left\{-\frac{1}{2} \le x \le \frac{1}{2}\right\} \\
R_2 \cap \left\{t=\frac{1}{2}\right\} &= \left\{-\frac{1}{2} \le x \le \frac{3}{2} \quad \text{and} \quad x \ge \frac{1}{2}\right\} = \left\{\frac{1}{2} \le x \le \frac{3}{2}\right\} \\
R_3 \cap \left\{t=\frac{1}{2}\right\} &= \left\{x \le -\frac{1}{2} \quad \text{and} \quad -\frac{3}{2} \le x \le \frac{1}{2}\right\} = \left\{-\frac{3}{2} \le x \le -\frac{1}{2}\right\} \\
R_4 \cap \left\{t=\frac{1}{2}\right\} &= \left\{x \le -\frac{1}{2} \quad \text{and} \quad x \ge \frac{1}{2}\right\} = \varnothing \\
R_5 \cap \left\{t=\frac{1}{2}\right\} &= \left\{x \le -\frac{3}{2}\right\}, \\
R_6 \cap \left\{t=\frac{1}{2}\right\} &= \left\{x \ge \frac{3}{2}\right\}
\end{aligned} \tag{32}$$

and (30) becomes

$$\begin{aligned}
u\left(x,\frac{1}{2}\right) &= \begin{cases} 1-\frac{1}{2}\left(\left|x-\frac{1}{2}\right| + \left|x+\frac{1}{2}\right|\right), & -\frac{1}{2} \le x \le \frac{1}{2} \\ \frac{1}{2}-\frac{1}{2}\left|x-\frac{1}{2}\right|, & \frac{1}{2} \le x \le \frac{3}{2} \\ \frac{1}{2}-\frac{1}{2}\left|x+\frac{1}{2}\right|, & -\frac{3}{2} \le x \le -\frac{1}{2} \\ 0, & |x| \ge \frac{3}{2} \end{cases} \\
&= \begin{cases} \frac{1}{2}, & -\frac{1}{2} \le x \le \frac{1}{2} \\ \frac{3}{4}-\frac{1}{2}x, & \frac{1}{2} \le x \le \frac{3}{2} \\ \frac{3}{4}+\frac{1}{2}x, & -\frac{3}{2} \le x \le -\frac{1}{2} \\ 0, & |x| \ge \frac{3}{2} \end{cases}
\end{aligned}$$

At $t=1$,

$$\begin{aligned}
R_1 \cap \{t=1\} &= \{0 \le x \le 2 \quad \text{and} \quad -2 \le x \le 0\} = \{x=0\} \\
R_2 \cap \{t=1\} &= \{0 \le x \le 2 \quad \text{and} \quad x \ge 0\} = \{0 \le x \le 2\} \\
R_3 \cap \{t=1\} &= \{x \le 0 \quad \text{and} \quad -2 \le x \le 0\} = \{-2 \le x \le 0\} \\
R_4 \cap \{t=1\} &= \{x \le 0 \quad \text{and} \quad x \ge 0\} = \{x=0\} \\
R_5 \cap \{t=1\} &= \{x \le -2\}, \\
R_6 \cap \{t=1\} &= \{x \ge 2\}
\end{aligned} \tag{33}$$

and (30) becomes

$$u(x,1) = \begin{cases} \frac{1}{2} - \frac{1}{2}|x-1|, & 0 \le x \le 2 \\ \frac{1}{2} - \frac{1}{2}|x+1|, & -2 \le x \le 0 \\ 0, & |x| \ge 2 \end{cases}$$

At $t = 3/2$,

$$\begin{aligned}
R_1 \cap \left\{t = \frac{3}{2}\right\} &= \left\{\frac{1}{2} \le x \le \frac{5}{2} \quad \text{and} \quad -\frac{5}{2} \le x \le -\frac{1}{2}\right\} = \varnothing \\
R_2 \cap \left\{t = \frac{3}{2}\right\} &= \left\{\frac{1}{2} \le x \le \frac{5}{2} \quad \text{and} \quad x \ge -\frac{1}{2}\right\} = \left\{\frac{1}{2} \le x \le \frac{5}{2}\right\} \\
R_3 \cap \left\{t = \frac{3}{2}\right\} &= \left\{x \le \frac{1}{2} \quad \text{and} \quad -\frac{5}{2} \le x \le -\frac{1}{2}\right\} = \left\{-\frac{5}{2} \le x \le -\frac{1}{2}\right\} \\
R_4 \cap \left\{t = \frac{3}{2}\right\} &= \left\{x \le \frac{1}{2} \quad \text{and} \quad x \ge -\frac{1}{2}\right\} = \left\{-\frac{1}{2} \le x \le \frac{1}{2}\right\} \\
R_5 \cap \left\{t = \frac{3}{2}\right\} &= \left\{x \le -\frac{5}{2}\right\}, \\
R_6 \cap \left\{t = \frac{3}{2}\right\} &= \left\{x \ge \frac{5}{2}\right\}
\end{aligned} \tag{34}$$

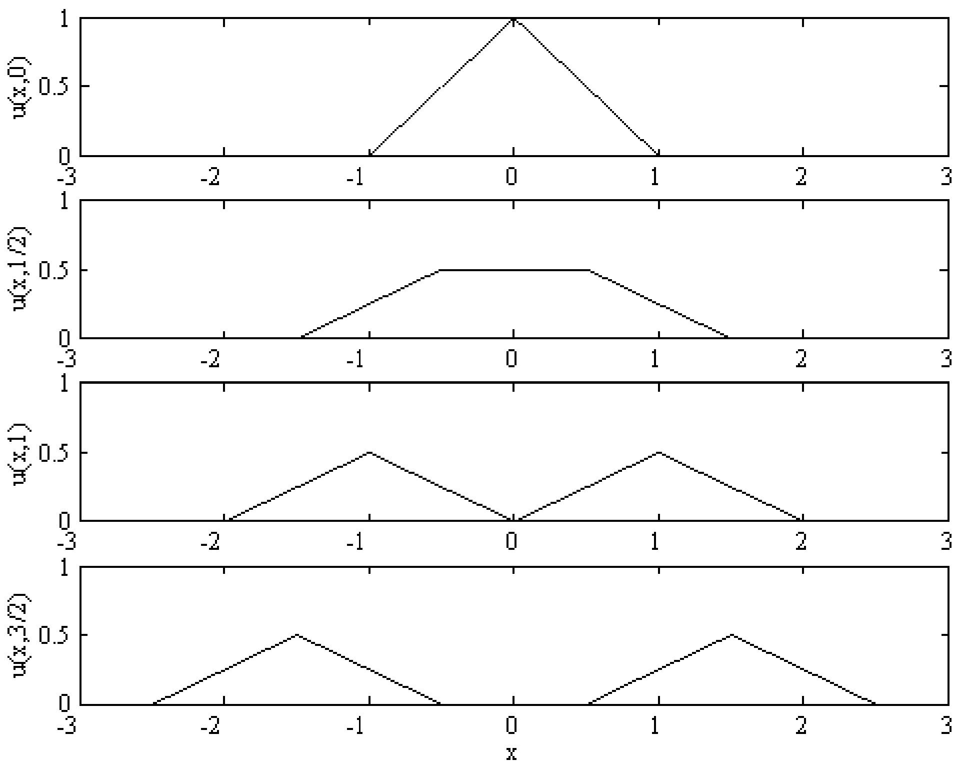
and (30) becomes

$$u\left(x, \frac{3}{2}\right) = \begin{cases} \frac{1}{2} - \frac{1}{2}|x-t|, & \frac{1}{2} \le x \le \frac{5}{2} \\ \frac{1}{2} - \frac{1}{2}|x+t|, & -\frac{5}{2} \le x \le -\frac{1}{2} \\ 0, & -\frac{1}{2} \le x \le \frac{1}{2} \\ 0. & |x| \ge 5/2 \end{cases}$$

These are plotted below.

(ii) Repeat the procedure in (i) for a string that has zero initial displacement but is given an initial velocity

$$u_t(x,0) = g(x) = \begin{cases} 2, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$$



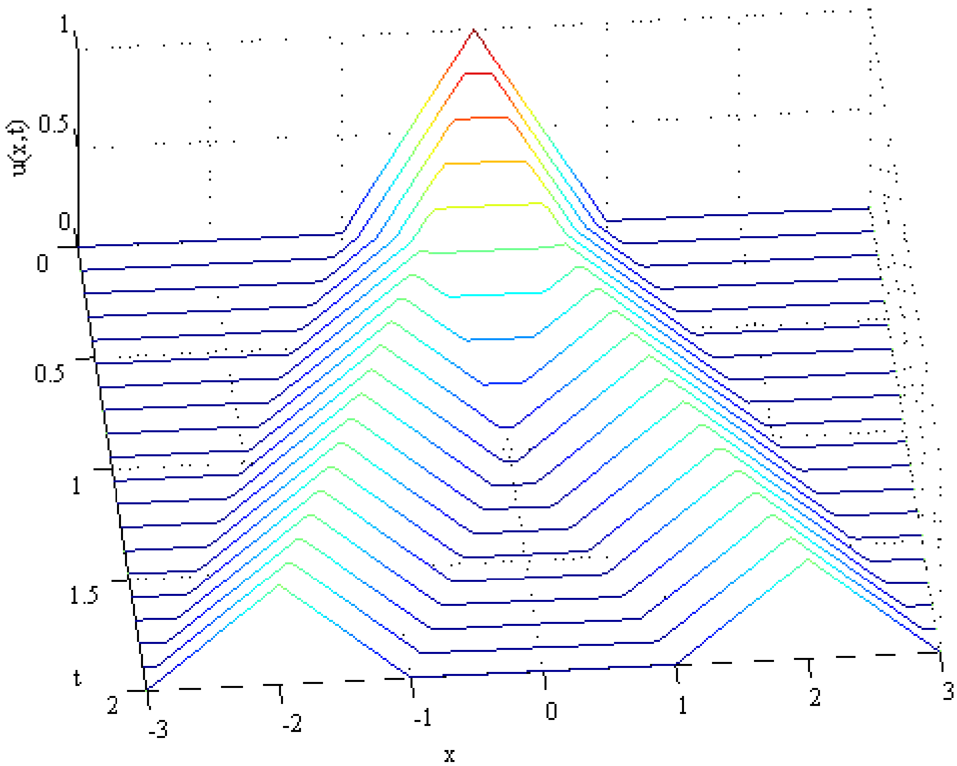
Solution: Step 1. Since $u(x,0) = 0$, D'Alembert's solution is

$$u(x,t) = \frac{1}{2}\int_{x-t}^{x+t} g(s)\, ds.$$

Step 2. Regions. Since $g(x)$ has the same form as $f(x)$, namely it is a case function of two cases, one for $|x| \leq 1$ and one for $|x| > 1$, the characteristic plot is the same as for (i) and the regions R_n are also the same (see (29)).

Step 3. Calculate solution in each interval. The integral can be calculated in each region, since each region tells us where $x + t$ and $x - t$ are with respect to ± 1. In region R_1, $-1 \leq x \pm t \leq 1$ so that

$$u(x,t) = \frac{1}{2}\int_{x-t}^{x+t} g(s)\, ds = \frac{1}{2}\int_{x-t}^{x+t} 2ds = 2t$$



In region R_2, $-1 \leq x - t \leq 1$ and $x + t \geq 1$ so that

$$\begin{aligned} u(x,t) &= \frac{1}{2}\int_{x-t}^{x+t} g(s)\,ds \\ &= \frac{1}{2}\int_{x-t}^{1} g(s)\,ds + \frac{1}{2}\int_{1}^{x+t} g(s)\,ds \\ &= \frac{1}{2}\int_{x-t}^{1} 2ds + \frac{1}{2}\int_{1}^{x+t} 0ds \\ &= 1 - (x - t) \end{aligned}$$

Similarly, in region R_3,

$$u(x,t) = \frac{1}{2}\int_{x-t}^{-1} g(s)\,ds + \frac{1}{2}\int_{-1}^{x+t} g(s)\,ds = \frac{1}{2}\int_{-1}^{x+t} 2ds = x + t + 1$$

In region R_4,

$$\begin{aligned} u(x,t) &= \frac{1}{2}\int_{x-t}^{-1} g(s)\,ds + \frac{1}{2}\int_{-1}^{1} g(s)\,ds + \frac{1}{2}\int_{1}^{x+t} g(s)\,ds \\ &= \frac{1}{2}\int_{-1}^{1} 2ds = 2 \end{aligned}$$

In regions R_5 and R_6, $u(x,t)=0$. To summarize,

$$u(x,t)=\frac{1}{2}\int_{x-t}^{x+t} g(s)\,ds=\begin{cases} 2t, & (x,t)\in R_1 \\ 1-x+t, & (x,t)\in R_2 \\ 1+x+t, & (x,t)\in R_3 \\ 2, & (x,t)\in R_4 \\ 0, & (x,t)\in R_5, R_6 \end{cases} \tag{35}$$

Step 4. For each specific time $t=t_0$, write the x-intervals corresponding to the sets R_n (i.e. the intersection of the set R_n with $\{t=t_0\}$, or in the figure above, where the line $t=t_0$ intersects the region R_n). Since the characteristics are the same, the intervals for x are the same for each time. As a check, we note that at $t=0$, the x-intervals correponding to each region are given by (31) and (35) becomes

$$u(x,0)=\frac{1}{2}\int_{x}^{x} g(s)\,ds=0=\begin{cases} 0, & |x|\leq 1 \\ 1-x, & x=1 \\ 1+x, & x=-1 \\ 2, & \varnothing \\ 0, & |x|\geq 1 \end{cases}=0=f(x)$$

so this checks! At $t=1/2$, the regions R_n are given by (32) and (35) becomes

$$u\left(x,\frac{1}{2}\right)=\frac{1}{2}\int_{x-1/2}^{x+1/2} g(s)\,ds=\begin{cases} 2\times\frac{1}{2}, & -\frac{1}{2}\leq x\leq\frac{1}{2} \\ 1-x+\frac{1}{2}, & \frac{1}{2}\leq x\leq\frac{3}{2} \\ 1+x+\frac{1}{2}, & -\frac{3}{2}\leq x\leq-\frac{1}{2} \\ 2, & \varnothing \\ 0, & |x|\geq\frac{3}{2} \end{cases}=\begin{cases} 1, & -\frac{1}{2}\leq x\leq\frac{1}{2} \\ \frac{3}{2}-x, & \frac{1}{2}\leq x\leq\frac{3}{2} \\ \frac{3}{2}+x, & -\frac{3}{2}\leq x\leq-\frac{1}{2} \\ 0, & |x|\geq\frac{3}{2} \end{cases}$$

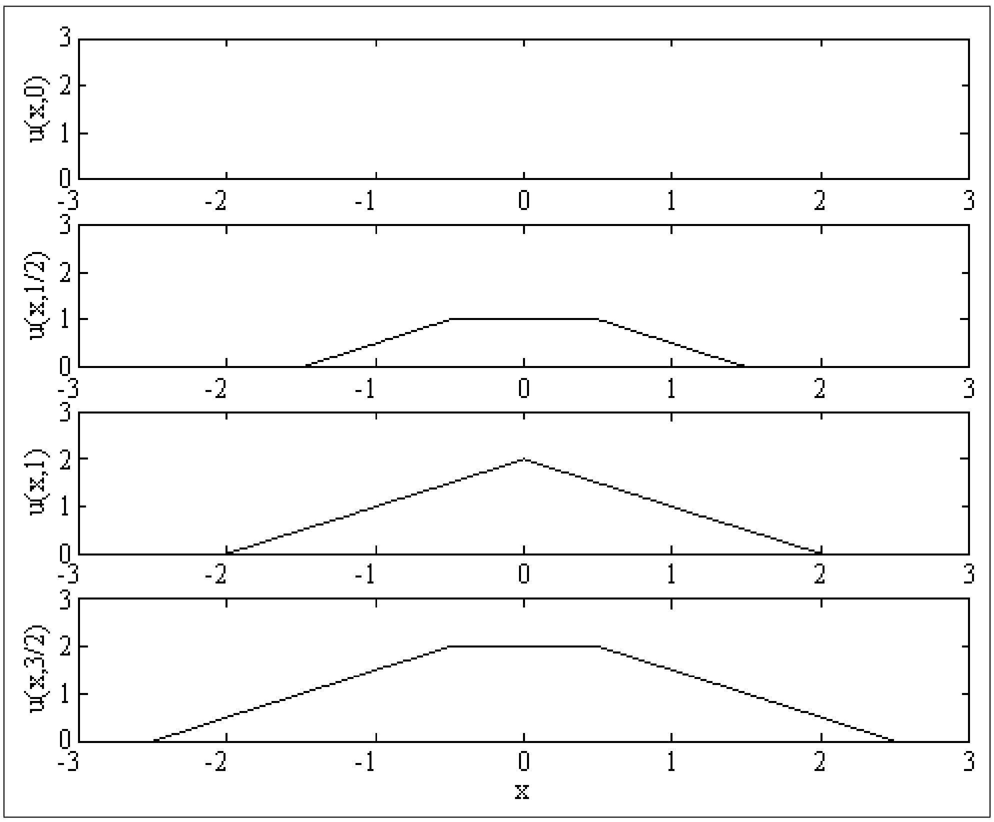
At $t=1$, the regions R_n are given by (33) and (35) becomes

$$u(x,1)=\frac{1}{2}\int_{x-1}^{x+1} g(s)\,ds=\begin{cases} 2-x, & 0\leq x\leq 2 \\ 2+x, & -2\leq x\leq 0 \\ 0, & |x|\geq 2 \end{cases}$$

At $t=3/2$, the regions R_n are given by (34) and (35) becomes

$$u\left(x,\frac{3}{2}\right)=\frac{1}{2}\int_{x-3/2}^{x+3/2} g(s)\,ds=\begin{cases} \frac{5}{2}-x, & \frac{1}{2}\leq x\leq\frac{5}{2} \\ \frac{5}{2}+x, & -\frac{5}{2}\leq x\leq-\frac{1}{2} \\ 2, & -\frac{1}{2}\leq x\leq\frac{1}{2} \\ 0, & |x|\geq\frac{5}{2} \end{cases}$$

These are plotted below.


u(x,0)
u(x,1/2)
u(x,1)
u(x,3/2)
x
-3
-2
-1
0
1
2
3

4 Problem 4

(i) For an infinite string (i.e. we don't worry about boundary conditions), what initial conditions would give rise to a purely forward wave? Express your answer in terms of the initial displacement $u(x,0) = f(x)$ and initial velocity $u_t(x,0) = g(x)$ and their derivatives $f'(x)$, $g'(x)$. Interpret the result intuitively.

Solution: Recall in class that we write D'Alembert's solution as

$$u(x,t) = P(x-t) + Q(x+t) \tag{36}$$

where

$$Q(x) = \frac{1}{2}\left(f(x) + \int_0^x g(s)\,ds + Q(0) - P(0)\right) \tag{37}$$

$$P(x) = \frac{1}{2}\left(f(x) - \int_0^x g(s)\,ds - Q(0) + P(0)\right) \tag{38}$$

To only have a forward wave, we must have

$$Q(x) = const = q_1$$

Substituting (37) gives

$$Q(x) = q_1 = \frac{1}{2}\left(f(x) + \int_0^x g(s)\,ds\right)$$

Differentiating in x gives

$$0 = \frac{1}{2}\left(\frac{df}{dx} + g(x)\right)$$

Thus

$$g(x) = -\frac{df}{dx} \tag{39}$$

Substituting (39) into (37) gives

$$Q(x) = \frac{1}{2}(f(0) + Q(0) - P(0))$$

and setting $x = 0$ yields $f(0) - P(0) = Q(0)$. Substituting this and (39) into (38) gives

$$P(x) = \frac{1}{2}(2f(x) - f(0) - Q(0) + P(0)) = f(x)$$

and hence

$$u(x,t) = f(x-t).$$

The displacement $u(x,t)$ only contains the forward wave! Intuitively, we have set the initial velocity of the string in such a way, given by Eq. (39), as to cancel the backward wave.

(ii) Again for an infinite string, suppose that $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$ are zero for $|x| > \varepsilon$. Prove that if $t + x > \varepsilon$ and $t - x > \varepsilon$, then the displacement $u(x,t)$ of the string is constant. Relate this constant to $g(x)$.

Solution: D'Alembert's solution for the wave equation is

$$u(x,t) = \frac{1}{2}(f(x-t) + f(x+t)) + \frac{1}{2}\int_{x-t}^{x+t} g(s)\, ds$$

If $x+t > \varepsilon$ and $t-x > \varepsilon$, then $|x+t| > \varepsilon$ and $|x-t| > \varepsilon$, so that $f(x \pm t) = 0$. Furthermore, $x - t < -\varepsilon$, so that

$$\int_{x-t}^{x+t} g(s)\, ds = \int_{-\varepsilon}^{\varepsilon} g(s)\, ds = \int_{-\infty}^{\infty} g(s)\, ds = c_\varepsilon$$

Thus c_ε is just the area under the curve $g(x)$, and

$$u(x,t) = c_\varepsilon, \qquad x + t > \varepsilon, \qquad t - x > \varepsilon.$$