Solutions for Problems for The 1-D Heat Equation

18.303 Linear Partial Differential Equations

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1 Problem 2

Find the Fourier sine and cosine series of

$$f(x) = \frac{1}{2}(1-x), \qquad 0 < x < 1.$$

a. State a theorem which proves convergence of each series. Graph the functions to which they converge.

Solution: The sine series is

$$\hat{f}(x) = \sum_{n=1}^{\infty} B_n \sin(n\pi x)$$

where

$$\begin{aligned} B_n &= 2\int_0^1 f(x)\sin(n\pi x)\,dx = \int_0^1 (1-x)\sin(n\pi x)\,dx \\ &= \left[\frac{-(1-x)\cos n\pi x}{n\pi} - \frac{\sin n\pi x}{(n\pi)^2}\right]_{x=0}^{1} \\ &= \frac{1}{n\pi} \end{aligned}$$

The cosine series is

$$\tilde{f}(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\pi x)$$

where

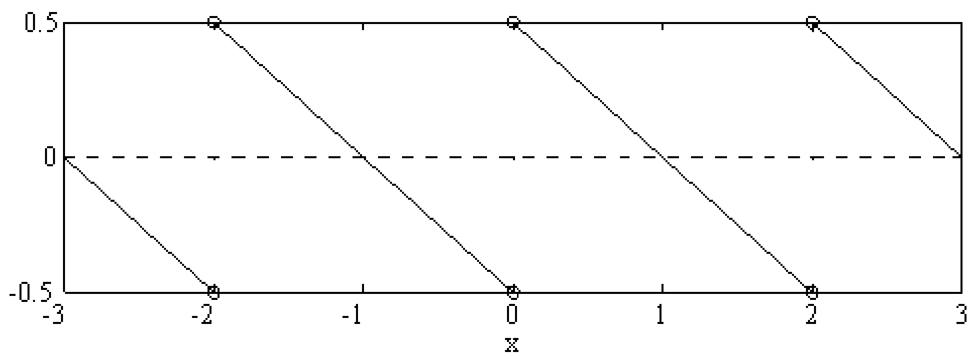
$$A_0 = \int_0^1 f(x)\,dx = \frac{1}{2}\int_0^1 (1-x)\,dx = \frac{1}{4}$$

$$
\begin{aligned}
A_n &= 2\int_0^1 f(x)\cos(n\pi x)\,dx = \int_0^1 (1-x)\cos(n\pi x)\,dx \\
&= \left[\frac{-(1-x)\sin n\pi x}{n\pi} - \frac{\cos n\pi x}{(n\pi)^2}\right]_{x=0}^{1} \\
&= \frac{1-\cos n\pi}{(n\pi)^2}
\end{aligned}
$$

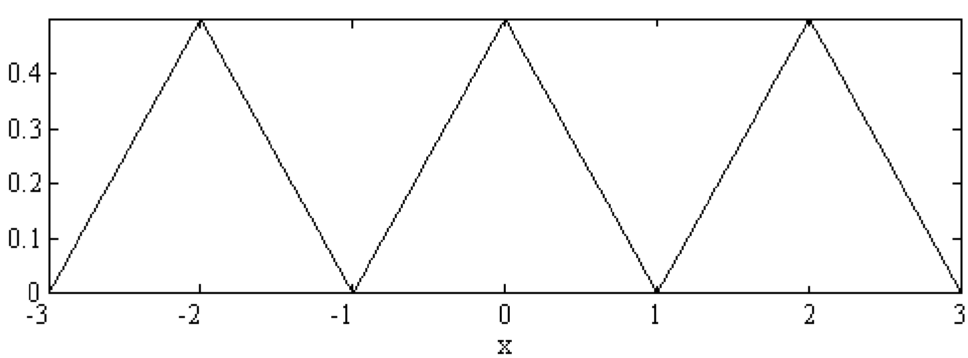
Thus $A_{2n} = 0$ and $A_{2n-1} = 2/\left((2n-1)^2\,\pi^2\right)$.

Both the sine and cosine series of $f(x)$ converge on the closed interval $[0,1]$ since $f(x)$ is piecewise continuous on $0 \le x \le 1$ and continuous on $0 < x < 1$, as required by the theorem in the notes.

The sine series is the odd periodic extension of $f(x)$, it is even, 2-periodic and discontinuous.



The cosine series is the even periodic extension of $f(x)$, it is even, 2-periodic and continuous.



b. Show that the Fourier sine series cannot be differentiated termwise (term-by-term). Show that the Fourier cosine series converges uniformly.

Solution: Differentiating $f(x)$ gives

$$\frac{df}{dx} = -\frac{1}{2}$$

Differentiating the sine series $\hat{f}(x)$ term-by-term gives

$$\frac{d\hat{f}}{dx} = \sum_{n=1}^{\infty} \cos(n\pi x)$$

This series does not converge because the summands do not approach zero as $n \to \infty$, for any x. For a series $\sum_n a_n$ to converge, the n'th summand a_n must approach zero as $n \to \infty$. An alternative method to show this series does not converge is to choose a single x where the series does not converge. Consider $x = 1/2$, then

$$\frac{d\hat{f}}{dx}\left(\frac{1}{2}\right) = \sum_{n=1}^{\infty} \cos\left(\frac{n\pi}{2}\right) = \sum_{,m=1}^{\infty} \cos(m\pi) = \sum_{,m=1}^{\infty} (-1)^m$$

where we let $m = 2n$. The partial sums

$$\sum_{,m=1}^{M} (-1)^m = \begin{cases} 0, & M \text{ even} \\ -1, & M \text{ odd} \end{cases}$$

do not converge, and hence the series at $x = 1/2$ does not converge. In particular, the term-by-term differentiated sine series does not converge, and hence the since series of $f(x)$, i.e. $\hat{f}(x)$ cannot be differentiated term-by-term.

To show the cosine series $\tilde{f}(x)$ converges uniformly, we apply the Weirstrass M-Test:

$$\left|\sum_{n=1}^{\infty} A_n \cos(n\pi x)\right| \le \sum_{n=1}^{\infty} |A_n \cos(n\pi x)| \le \sum_{n=1}^{\infty} |A_n| = \sum_{n=1}^{\infty} \frac{2}{(2n-1)^2 \pi^2} \le \frac{2}{\pi^2} \sum_{m=1}^{\infty} \frac{1}{m^2}$$

We know $\sum_{m=1}^{\infty} \frac{1}{m^2}$ converges from our class notes, and hence by the Weirstrass M-Test, the cosine series $\tilde{f}(x)$ converges uniformly.

2 Problem 3

A bar with initial temperature profile $f(x) > 0$, with ends held at 0^o C, will cool as $t \to \infty$, and approach a steady-state temperature 0^oC. However, whether or not all parts of the bar start cooling initially depends on the shape of the initial temperature profile. The following example may enable you to discover the relationship.

a. Find an initial temperature profile $f(x)$, $0 \le x \le 1$, which is a linear combination of $\sin \pi x$ and $\sin 3\pi x$, and satisfies $\frac{df}{dx}(0) = 0 = \frac{df}{dx}(1)$, $f\left(\frac{1}{2}\right) = 4$.

Solution: A linear combination of $\sin \pi x$ and $\sin 3\pi x$ is

$$f(x) = a \sin 3\pi x + b \sin \pi x$$

Imposing the conditions gives

$$\begin{aligned} 0 &= \frac{df}{dx}(0) = \pi(3a+b) \\ 0 &= \frac{df}{dx}(1) = -\pi(3a+b) \\ 4 &= f\left(\frac{1}{2}\right) = -a+b \end{aligned}$$

The first two equation are redundant. Solving for a, b gives

$$a = -1, \qquad b = 3.$$

Thus

$$f(x) = -\sin 3\pi x + 3 \sin \pi x \tag{1}$$

b. Solve the problem

$$u_t = u_{xx}; \qquad u(0,t) = 0 = u(1,t); \qquad u(x,0) = f(x).$$

This is easy, you can just write down the solution we had in class - but make sure you know how to get it.

Solution: This is the basic heat problem we considered in class, with solution

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin(n\pi x) e^{-n^2\pi^2 t} \tag{2}$$

where

$$B_n = 2\int_0^1 f(x) \sin(n\pi x)\, dx \tag{3}$$

and $f(x)$ is given in (1). The form of (1) is already a sine series, with $B_1 = 3$, $B_3 = -1$ and $B_n = 0$ for all other n. You can check this for yourself by computing integrals in (3) for $f(x)$ given by (1), from the orthogonality of $\sin n\pi x$. Therefore,

$$u(x,t) = 3\sin(\pi x) e^{-\pi^2 t} - \sin(3\pi x) e^{-9\pi^2 t} \tag{4}$$

c. Show that for some x, $0 \leq x \leq 1$, $u_t(x, 0)$ is positive and for others it is negative. How is the sign of $u_t(x, 0)$ related to the shape of the initial temperature profile? How is the sign of $u_t(x, t)$, $t > 0$, related to subsequent temperature profiles? Graph the temperature profile for $t = 0, 0.2, 0.5, 1$ on the same axis (you may use Matlab).

Differentiating $u(x, t)$ in time gives

$$u_t(x,t) = -\pi^2 \left(3 \sin(\pi x) e^{-\pi^2 t} - 9 \sin(3\pi x) e^{-9\pi^2 t}\right)$$

Setting $t = 0$ gives

$$u_t(x, 0) = -3\pi^2 (\sin(\pi x) - 3 \sin(3\pi x))$$

Note that

$$u_t\left(\frac{1}{6}, 0\right) = \frac{15}{2}\pi^2 > 0, \qquad u_t\left(\frac{1}{2}, 0\right) = -12\pi^2 < 0$$

Thus at $x = 1/6$, u_t is positive and for $x = 1/2$, u_t is negative.

From the PDE,

$$u_t = u_{xx}$$

and hence the sign of u_t gives the concavity of the temperature profile $u(x, t_0)$, t_0 constant. Note that for $u_{xx}(x, t_0) > 0$, the profile $u(x, t_0)$ is concave up, and for $u_{xx}(x, t_0) < 0$, the profile $u(x, t_0)$ is concave down. At $t_0 = 0$, the sign of $u_t(x, 0)$ give the concavity of the initial temperature profile $u(x, 0) = f(x)$.

The plots of $u(x, t_0)$ for $t_0 = 0, 0.2, 0.5, 1$ are below.