Problems for the 1-D Wave Equation

18.303 Linear Partial Differential Equations

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1 Problem 1

(i) Generalize the derivation of the wave equation where the string is subject to a damping force $-b\partial u/\partial t$ per unit length to obtain

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} - 2k\frac{\partial u}{\partial t} \tag{1}$$

All variables will be left in dimensional form in this problem to make things a little different. How is the constant k related to b? What are the dimensions of b and k? The constant 2 is included for later convenience.

(ii) Use separation of variables to find the normal modes of the damped Wave Equation (1) subject to the BCs

$$u(0,t) = 0 = u(l,t)$$

Impose a restriction on the parameters c, l, k which will guarantee that all solutions are oscillatory in time. You may assume that the eigenvalues and eigenfunctions are

$$\lambda_n = \frac{n^2\pi^2}{l^2}, \qquad X_n(x) = \sin\frac{n\pi x}{l}, \qquad n = 1, 2, 3...$$

(iii) Express the frequency $\widetilde{f}_n$ of the oscillatory part of the n'th normal mode in terms of the frequency of the undamped mode $f_n = nc/(2l)$. What difference does the damping make?

(iv) Show that the solution of the damped wave equation subject to the BCs (1) and the initial condition

$$u(x,0) = f(x), \qquad \frac{\partial u}{\partial t}(x,0) = 0$$

is given by

$$u(x,t) = e^{-kt} \sum_{n=1}^{\infty} \left(\alpha_n \cos\left(2\pi \widetilde{f}_n t\right) + \beta_n \sin\left(2\pi \widetilde{f}_n t\right)\right) \sin\left(\frac{n\pi x}{l}\right)$$

Express the constants α_n, β_n in terms of the Fourier Sine coefficients B_n of f.

2 Problem 2

Prove that if a vibrating string is damped, i.e. subject to the PDE in Problem 1(i), then the energy $E(t)$ is monotone decreasing. You may use the formula we derived in lecture,

$$E(t) = \frac{\rho}{2} \int_0^l \left(u_t^2 + c^2 u_x^2\right) dx$$

3 Problem 3

(i) Suppose that an "infinite string" has an initially displacement

$$u(x,0) = f(x) = \begin{cases} x+1, & -1 \le x \le 0 \\ -x+1, & 0 \le x \le 1 \\ 0, & |x| > 1 \end{cases}$$

and zero initial velocity $u_t(x,0) = 0$. Write down the solution of the wave equation

$$u_{tt} = u_{xx}$$

with ICs $u(x,0) = f(x)$ and $u_t(x,0) = 0$ using D'Alembert's formula. Illustrate the nature of the solution by sketching the ux-profiles $y = u(x,t)$ of the string displacement for $t = 0, 1/2, 1, 3/2$.

(ii) Repeat the procedure in (i) for a string that has zero initial displacement but is given an initial velocity

$$u_t(x,0) = g(x) = \begin{cases} 2, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$$

4 Problem 4

(i) For an infinite string (i.e. we don't worry about boundary conditions), what initial conditions would give rise to a purely forward wave? Express your answer in terms of the initial displacement $u(x,0) = f(x)$ and initial velocity $u_t(x,0) = g(x)$ and their derivatives $f'(x)$, $g'(x)$. Interpret the result intuitively.

(ii) Again for an infinite string, suppose that $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$ are zero for $|x| > \varepsilon$. Prove that if $t + x > \varepsilon$ and $t - x > \varepsilon$, then the displacement $u(x,t)$ of the string is constant. Relate this constant to $g(x)$.

5 Problem 5

(i) Let $\bar{g} : \mathbb{R} \to \mathbb{R}$ be the odd periodic extension of $g : [0,1] \to \mathbb{R}$, where g is smooth ($g'(x)$ is continuous). Verify that

$$u(x,t) = \frac{1}{2} \int_{x-t}^{x+t} \bar{g}(s)\, ds, \qquad 0 \le x \le 1, \qquad t \ge 0$$

is a solution of the vibrating string problem

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= \frac{\partial^2 u}{\partial x^2}; \qquad u(0,t) = 0 = u(1,t), \qquad t \ge 0 \\ u(x,0) &= 0, \qquad \frac{\partial u}{\partial x}(x,0) = g(x), \qquad 0 \le x \le 1. \end{aligned}$$

(ii) If $g(x) = 2\varepsilon x(1-x)$, $0 \le x \le 1$, find the displacement of the string $u(x,t)$ at $x = 1/4$ when $t = 3/2$.

6 Problem 6

Consider a semi-infinite vibrating string. The vertical displacement $u(x,t)$ satisfies

$$\begin{aligned} u_{tt} &= u_{xx}, \qquad x \ge 0, \qquad t \ge 0 \\ u(0,t) &= 0, \qquad t \ge 0 \\ u(x,0) &= f(x), \qquad u_t(x,0) = g(x), \qquad x \ge 0. \end{aligned} \tag{2}$$

(a) Show that D'Alembert's formula solves (2) when $f(x)$ and $g(x)$ are extended to be odd functions.

(b) Let

$$f(x) = \begin{cases} \sin^2(\pi x), & 1 \le x \le 2 \\ 0, & 0 \le x \le 1, \quad x \ge 2 \end{cases}$$

and $g(x) = 0$ for $x \ge 0$. Sketch u vs. x for $t = 0, 1, 2, 3$.

7 Problem 7

The acoustic pressure in an organ pipe obeys the 1-D wave equation (in physical variables)

$$p_{tt} = c^2 p_{xx}$$

where c is the speed of sound in air. Each organ pipe is closed at one end and open at the other. At the closed end, the BC is that $p_x(0,t) = 0$, while at the open end, the BC is $p(l,t) = 0$, where l is the length of the pipe.

(a) Use separation of variables to find the normal modes $u_n(x,t)$.

(b) Give the frequencies of the normal modes and sketch the pressure distribution for the first two modes.

(c) Given initial conditions $p(x,0) = f(x)$ and $p_t(x,0) = g(x)$, write down the general initial boundary value problem (PDE, BCs, ICs) for the organ pipe and determine the series solutions.