Math 18.305 Fall 2004/05
Assignment 6: Boundary Layers
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1. Solve approximately the following two equations

   (a) $\epsilon y'' + (1+x)^2 y' + y = 0$, $0 < x < 1$, with $y(0) = 0$ and $y(1) = 2$.

   (b) $\epsilon y'' - (1+x^2) y' + y = 0$, $0 < x < 1$, with $y(0) = 0$ and $y(1) = 2$

Solutions:

1. Solve approximately the following two equations

   In both problems, the highest derivative is multiplied by a perturbation parameter ϵ, and hence the problems are solved using boundary layer techniques.

   (a) Since $a(x) = (1+x)^2 > 0$, the rapidly varying solution is decreasing and thus the boundary layer is at $x = 0$, and it has width ϵ.

We can find the solution outside the boundary layer by solving

$$(1+x)^2 y'_{out} + y_{out} = 0$$

which gives

$$y_{out} = C_1 e^{\frac{1}{1+x}}$$

where C_1 is a constant. Since the rapidly varying function is negligible at $x = 1$, we can use the boundary condition at $x = 0$ to determine the value of C_1 :

$$y_{out}(1) = C_1 e^{1/2} = 2$$

thus $C_1 = 2e^{-1/2}$. To find the behaviour of the rapidly varying function near $x = 0$, we look at

$$\epsilon y'' + (1+0)^2 y' = 0$$

which gives

$$y_r \approx C_2 e^{-x/\epsilon}$$

Using the boundary condition at $x = 0$, i.e

$$C_2 + y_{out}(0) = 0$$

we find $C_2 = -2e^{-1/2}$. Hence the solution is

$$y_{uniform} = 2e^{-1/2} e^{\frac{1}{1+x}} - 2e^{-1/2} e^{-x/\epsilon}$$

1. **(b)** Since $a(x) = -(1+x^2) < 0$, the rapidly varying solution is increasing and thus the boundary layer is at $x = 1$, and it has width ϵ.

We can find the solution outside the boundary layer by solving

$$-(1+x^2)y'_{out} + y_{out} = 0$$

which gives

$$y_{out} = C_1 e^{\tan^{-1} x}$$

where C_1 is a constant. Since the rapidly varying function is negligible at $x = 0$, we can use the boundary condition at $x = 0$ to determine the value of C_1 :

$$y_{out}(0) = C_1 e^0 = 0$$

thus $C_1 = 0$. Hence $y_{out} \equiv 0$.Therefore the solution is approximately zero outside the boundary layer.
To find the behaviour of the rapidly varying function near $x = 1$, we look at

$$\epsilon y'' - (1+1^2)y' = 0$$

which gives

$$y_r \approx C_2 e^{2(x-1)/\epsilon}$$

Using the boundary condition at $x = 1$, we find $C_2 = 2$. Hence one may think that the solution can be approximated by

$$y_{uniform} = y_r + y_{out} = y_r = 2e^{2(x-1)/\epsilon} \tag{1}$$

That is correct in that the absolute error between the exact solution and $y_{uniform}$ given by (1) will be small. But since the solution itself is also "small", the relative error may be large. In such a case, it may be useful to calculate higher order approximations to y_r which, in the present case, equals $y_{uniform}$.
To calculate a better approximation, we let

$$y_r = e^{\frac{1}{\epsilon}\int_1^x (1+x^2)dx} v(x) = e^{\frac{1}{\epsilon}(x+\frac{1}{3}x^3-\frac{4}{3})dx} v(x)$$

where $v(x)$ is expected not to be rapidly varying. Substituting this into the differential equation, we obtain

$$v' - \frac{2x+1}{1+x^2} v = \epsilon v''$$

Neglecting the right hand side, we find

$$\ln v = -\ln(1+x^2) - \tan^{-1} x + c$$

or

$$v(x) = \frac{2C}{1+x^2} e^{-(\tan^{-1} x - \frac{\pi}{4})}$$

where c and C are some constants. Using the boundary condition at $x = 1$, we see that $C = 2$. Hence we find

$$y_{uniform} = y_{out} + y_r = y_r = \frac{4}{1+x^2} e^{-(\tan^{-1} x - \frac{\pi}{4})} e^{\frac{1}{\epsilon}(x+\frac{1}{3}x^3-\frac{4}{3})dx}$$

which is more accurate than (1).