

Lecture 18

Interior Turning Points

In this lecture we'll examine some examples in which there is a turning point inside the interval where (9.13) holds. This is to say that the turning point is an interior point.

This is a good time to make a subtle elaboration on how to continue a solution through a turning points. We shall put, without loss of generality, the turning point at the origin.

Let's first consider the case in which $\alpha > 0$. The rapidly varying solution is increasing for $x < 0$ and is decreasing for $x > 0$. Let us start with the rapidly varying solution (9.24), at the lower endpoint $x = -1$ and continue this solution to the right. Before we reach the origin, indeed at a distance $\sqrt{\epsilon}$ away from the origin, (9.24) fails to be a good approximation. To continue this solution through the origin, we must join it with $y_2(x)$ which, by the second asymptotic form of (9.52), is rapidly varying when x is negative and is with a magnitude much larger than $\sqrt{\epsilon}$. As we continue this solution to positive values of x , it turns into the slowly varying solution when $x \gg \sqrt{\epsilon}$, as can be verified with the use of the first asymptotic formula of (9.52). We mention that, for $\alpha > 0$, it is not meaningful to start with the slowly varying solution at $x = -1$ and continue it to the right. To see this, consider the solution near $x = -1$ which is a sum of a slowly varying solution and a rapidly varying solution. Let the latter being very small compared to the former at $x = -1$. As we continue this solution to the right, the rapidly solution grows exponentially and may become comparable to or even larger than the slowly varying solution. This says that if we start at $x = -1$, it is difficult to distinguish a slowly varying solution with the sum of a slowly varying solution and a rapidly varying solution.

Similarly, in order to obtain the second independent solution of the differential equation, we start with the rapidly varying solution at the upper endpoint $x = 1$ and continue it to the left. Inside the boundary layer, it is equal to $y_1(x)$, as we can see from the first asymptotic formula of (9.51). As we continue this solution to the region $x < 0$, it turns into the slowly varying solution, as we can see from the second asymptotic formula of (9.51).

To repeat, if α is positive, the rapidly varying solution decreases exponentially as one leaves the origin in either direction. Therefore, if a rapidly varying solution and a slowly varying solution are of the same order of magnitude at $x = 0$, then the former is much smaller than the latter as one leaves the origin. We shall call the former the small solution, and the latter the large solution. It is meaningful to take a small solution at one side of the turning point and continue it across the turning point. The small solution turns into the large solution at the other side of the turning point. But it is not meaningful to start with the large solution and continue it across the turning point.

If $\alpha < 0$, the rapidly varying solution is increasing in the region $x > 0$ and decreasing in the region $x < 0$. Thus the rapidly varying solution is the large solution and the slowly varying solution is the small solution. As before, we must start with the small solution and continue it across the turning. Thus we start with the slowly varying solution at an endpoint and continue it across the origin. This solution turns into the rapidly varying solution as one crosses the origin.

We shall give a couple of examples to illustrate how to use this concept of small solution and large solution to solve a boundary layer problem with an interior turning point. It is also possible to use this concept to solve boundary layer problems with two or more turning points. (See homework problem 12.)

Problem for the Reader:

Solve

$$\epsilon y'' + 2(\tan x)y' - y = 0, \quad -1 < x < 1,$$

with

$$y(-1) = 1, \text{ and } y(1) = 3.$$

Answer

We have

$$a(x) > 0, \quad x > 0; \quad a(x) < 0, \quad x < 0.$$

We see that the rapidly varying solution in the region $x > 0$ is decreasing with x . This means that there is no boundary layer at the upper endpoint $x = 1$. Also, the rapidly varying solution in the

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region $x < 0$ is increasing with x . Thus there is no boundary layer at the lower endpoint $x = -1$. The boundary layer is at the turning point $x = 0$. The width of the boundary layer near $x = 0$ is of the order of $\sqrt{\epsilon}$.

Note that, on either side of the origin, the rapidly varying solution is always the small solution, and the slowly varying solution is always the large solution.

The slowly varying solution is given by

$$2(\tan x)y'_s - y_s \simeq 0,$$

or

$$y_s(x) = c\sqrt{\sin x}.$$

Let us start with the small solution, which is the rapidly varying solution, at $x = -1$. After we continue this solution to the region $0 < x < 1$, it becomes the slowly varying solution given by

$$y_{out}^{(a)} = 3\sqrt{\frac{\sin x}{\sin 1}},$$

where the boundary condition $y(1) = 3$ has been taken into account. Since this solution is $O(1)$ in the region $x > 0$, it is exponentially small in the region $x < 0$.

When x is small and positive, we have

$$y_{out}^{(a)}(x) \approx 3\sqrt{\frac{x}{\sin 1}}.$$

This solution is a good approximation outside of the boundary layer near the origin of the width $\epsilon^{1/2}$. Since $y_{out}^{(a)}(\epsilon^{1/2})$ is of the order of $\epsilon^{1/4}$, we expect the value of the solution inside the boundary layer to be of the order of $\epsilon^{1/4}$, which is small.

Next we start with the rapidly varying solution at $x = 1$ and continue it to the region $-1 < x < 0$, where it becomes the slowly varying solution given by

$$y_{out}^{(b)} = \sqrt{-\sin x / \sin 1}.$$

When x is small and negative, we have

$$y_{out}^{(b)} \approx \sqrt{-x / \sin 1}.$$

For $x > 0$, this solution is exponentially small and can be ignored.

Let us find the solution $y_{in}(x)$ inside the boundary layer near $x = 0$. By (9.49) and (9.50), we have $X = \sqrt{\frac{2}{\epsilon}}x$ and

$$y_{in}(x) = e^{-x^2/(2\epsilon)} \left[c_1 D_{-3/2} \left(\sqrt{\frac{2}{\epsilon}}x \right) + c_2 D_{-3/2} \left(-\sqrt{\frac{2}{\epsilon}}x \right) \right].$$

As $X \rightarrow \infty$, we have

$$y_{in}(x) \rightarrow c_2 2^{7/4} \epsilon^{-1/4} \sqrt{x}.$$

Matching with $y_{out}^{(a)}$ in the region $x > 0$, we get

$$c_2 \approx 3\sqrt{\frac{1}{2^{7/2} \sin 1}} \epsilon^{1/4}.$$

As $X \rightarrow -\infty$, we have

$$y_{in}(x) \rightarrow c_1 2^{7/4} \epsilon^{-1/4} \sqrt{-x}.$$

Matching with $y_{out}^{(b)}(x)$, we get

$$c_1 \approx \sqrt{\frac{1}{2^{7/2} \sin 1}} \epsilon^{1/4}.$$

Thus

$$y_{in}(x) = e^{-x^2/(2\epsilon)} \sqrt{\frac{1}{2^{7/2} \sin 1}} \epsilon^{1/4} \left[D_{-3/2} \left(\sqrt{\frac{2}{\epsilon}}x \right) + 3D_{-3/2} \left(-\sqrt{\frac{2}{\epsilon}}x \right) \right],$$

which is of the order of $\epsilon^{1/4}$ when X is of the order of unity, as expected.