18.305 Solutions
Assignment 3: WKB method
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1. **(problem 4.a. on page 205) Solve the following equation in closed form**

$$y'' + x^m y = 0 \tag{1}$$

Solution:

For simplicity, let us first assume that we are looking for the solution for $x > 0$.The WKB solutions of (1) are easily found to be

$$y^{\pm}_{WKB} = x^{-m/4}\exp(\pm i\frac{2}{m+2}x^{\frac{m+2}{2}}) \tag{2}$$

As we know, the WKB solutions for the Bessel equation

$$[\rho^2\frac{d^2}{d\rho^2} + \rho\frac{d}{d\rho} + \rho^2 - p^2]Y(\rho) = 0$$

are

$$Y_{WKB} = \rho^{-1/2}\exp(\pm i\rho) \tag{3}$$

Comparing the exponent of the solution (2) with that of (3) suggests using the transformation

$$\rho = \frac{2}{m+2}x^{\frac{m+2}{2}}$$

With this identification of the independent variables, (2) will become

$$y^{\pm}_{WKB} = \rho^{-\frac{m}{2(m+2)}}\exp(\pm i\rho)$$

To make the WKB solutions exactly the same, we put

$$y = \rho^{-\frac{m}{2(m+2)}}\rho^{1/2}Y = \rho^{\frac{1}{m+2}}Y \tag{4}$$

We now plug

$$\begin{aligned}
\rho &= \frac{2}{m+2}x^{\frac{m+2}{2}} \\
x &= (\frac{m+2}{2}\rho)^{\frac{2}{m+2}} \\
\frac{d}{dx} &= \frac{d\rho}{dx}\frac{d}{d\rho} = -(\frac{m+2}{2}\rho)^{\frac{m}{m+2}}\frac{d}{d\rho} \\
\frac{d^2}{dx^2} &= [(\frac{m+2}{2}\rho)^{\frac{m}{m+2}}\frac{d}{d\rho}][(\frac{m+2}{2}\rho)^{\frac{m}{m+2}}\frac{d}{d\rho}] = (\frac{m+2}{2}\rho)^{\frac{2m}{m+2}}\frac{d^2}{d\rho^2} + (\frac{m+2}{2})^{\frac{2m}{m+2}}(\frac{m}{m+2})\rho^{\frac{-2}{m+2}}\frac{d}{d\rho}
\end{aligned}$$

into (1), to obtain

$$[\rho^2\frac{d^2}{d\rho^2} + \frac{m}{m+2}\rho\frac{d}{d\rho} + \rho^2]Y(\rho) = 0$$

Now we will use (4) on this last equation, we will make the replacements

$$\frac{d}{d\rho} \rightarrow (\frac{d}{d\rho} + \frac{1}{\rho(m+2)})$$
$$\frac{d^2}{d\rho^2} \rightarrow (\frac{d}{d\rho} + \frac{1}{\rho(m+2)})^2 = \frac{d^2}{d\rho^2} + \frac{2}{\rho(m+2)}\frac{d}{d\rho} + \frac{1}{\rho^2(m+2)^2} - \frac{1}{\rho^2(m+2)}$$

which gives us the equation

$$[\rho^2\frac{d^2}{d\rho^2} + \rho\frac{d}{d\rho} + \rho^2 - (\frac{1}{m+2})^2]Y(\rho) = 0$$

We observe that this is Bessel's equation with $p = \frac{1}{m+2}$. Therefore its general solution is

$$Y(\rho) = aJ_{\frac{1}{m+2}}(\rho) + bJ_{\frac{-1}{m+2}}(\rho)$$

where a and b are arbitrary constants. Therefore the general solution of our original equation (1) is

$$y = x^{1/2}[aJ_{\frac{1}{m+2}}(\frac{2}{m+2}x^{\frac{m+2}{2}}) + bJ_{\frac{-1}{m+2}}(\frac{2}{m+2}x^{\frac{m+2}{2}})]$$

which is obtained by undoing the variable transformations. Here a and b are again arbitrary constants.

This last equation is valid when $x > 0$ or m is even. If $x < 0$ and m is odd, we will have the form

$$y = |x|^{1/2}[aJ_{\frac{1}{m+2}}(\frac{2i}{m+2}|x|^{\frac{m+2}{2}}) + bJ_{\frac{-1}{m+2}}(\frac{2i}{m+2}|x|^{\frac{m+2}{2}})]$$

which can be obtained by following the same steps.

2. **(problem 5.b. on page 206)Obtain the WKB solutions of the following equation and determine for what values of $t > 0$ are these approximations good.**

$$\frac{d^2y}{dt^2} + e^{-\epsilon t}y = 0, \text{ where } 0 < \epsilon << 1.$$

Solution:

We let

$$p^2 = e^{-\epsilon t}$$
$$\Rightarrow \int pdt = -\frac{2}{\epsilon}e^{-\frac{1}{2}\epsilon t}$$

hence the WKB solutions are

$$\exp(\frac{1}{4}\epsilon t)\exp(\pm i\frac{2}{\epsilon}e^{-\frac{1}{2}\epsilon t}) = \exp(\frac{1}{4}\epsilon t \pm i\frac{2}{\epsilon}e^{-\frac{1}{2}\epsilon t})$$

which are "good" for

$$|\frac{d}{dt}\frac{1}{p}| = |\frac{d}{dt}e^{\frac{1}{2}\epsilon t}| = |\frac{\epsilon}{2}e^{\frac{1}{2}\epsilon t}| << 1$$

i.e for

$$t << \frac{2}{\epsilon}\ln(\frac{2}{\epsilon})$$

3. **Find the WKB solutions of the following equation and determine the values of x for which those WKB solutions are good approximations.**

$$xy'' + (c - x)y' - ay = 0$$

Solution:

We first rewrite the eqaution in the form

$$y'' + (\frac{c}{x} - 1)y' - \frac{a}{x}y = 0 \tag{5}$$

To be able to obtain the WKB solutions, we shall first transform the above equation into the form

$$y'' - \eta^2 y = 0$$

via a transformation. Let

$$y = e^{f(x)}Y$$

then

$$\begin{aligned} D &\rightarrow D + f' \\ D^2 &\rightarrow D^2 + 2f'D + (f')^2 + f'' \end{aligned}$$

and (5) becomes

$$[D^2 + (2f' + \frac{c}{x} - 1)D + (f')^2 + f'' + f' - \frac{a}{x}]Y = 0 \tag{6}$$

To make the coefficient of Y' zero, we must have

$$f' = -\frac{1}{2}(\frac{c}{x} - 1)$$

$$\begin{aligned} &\Rightarrow \quad f(x) = \frac{1}{2}x - \frac{1}{2}c\ln x \\ y &= x^{-\frac{c}{2}}e^{\frac{1}{2}x} \end{aligned}$$

Therefore (6) is

$$[D^2 - (\frac{1}{4} - \frac{c-2a}{2x} - \frac{1}{4}\frac{c^2 - 2c}{x^2})]Y = 0 \tag{7}$$

We will now find WKB solutions of this last differential equation.

$$\begin{aligned} \eta &= (\frac{1}{4} - \frac{c-2a}{2x} - \frac{1}{4}\frac{c^2}{x^2})^{1/2} = \frac{1}{2}(1 - 2\frac{c-2a}{x} - \frac{c^2-2c}{x^2})^{1/2} \\ &\approx \frac{1}{2}(1 - \frac{c-2a}{x} + \frac{1}{4}\frac{c-2a}{x^2} - \frac{1}{2}\frac{c^2-2c}{x^2} + ...) \\ &= \frac{1}{2}(1 - \frac{c-2a}{x} + O(\frac{1}{x^2})) \end{aligned}$$

The important thing to keep in mind is that we can neglect any terms in η which are smaller than $O(\frac{1}{x})$ but we cannot neglect terms of $O(\frac{1}{x})$ in η. This is because, when η is integrated, a term of $O(\frac{1}{x})$ will give a logarithmic function, and in the WKB solution this will show up as a factor in front. Smaller terms, however, will not matter. For example $\frac{1}{x^2}$ in η will integrate to $-\frac{1}{x}$, which will contribute a factor $\exp(-\frac{1}{x})$. As $x >> 1$, this factor will be very close to 1, hence negligible. Therefore, we can take

$$\begin{aligned} \eta &= \frac{1}{2} - \frac{c-2a}{2x} \\ \int \eta dx &= \frac{1}{2}x - (\frac{c}{2} - a)\ln x \end{aligned}$$

Also

$$\frac{1}{\sqrt{\eta}} = \frac{1}{\sqrt{\frac{1}{2} - \frac{c-2a}{2x}}} \approx \sqrt{2} : \text{just a constant}$$

because of the same type of reasons. Hence the WKB solutions of (7) are

$$x^{\mp(\frac{c}{2}-a)} \exp(\pm\frac{1}{2}x)$$

Hence the WKB solutions of the original problem are

$$x^{-a}, \ x^{a-c}e^x$$

which are familiar function from the 2nd homework assignment. For the validity issue we look at the quantity

$$|\frac{d}{dx}(\frac{1}{2} - \frac{c-2a}{2x})^{-1}| = \frac{\frac{c-2a}{2x^2}}{(\frac{1}{2} - \frac{c-2a}{2x})^2} << 1$$

which is satisfied when

$$x >> \max\{\sqrt{|\frac{c}{2} - a|}, |c-2a|\}$$

This last condition, combined with

$$x >> \sqrt{|c^2/2 - c|}$$

justify also the various approximations we made during the course of the solution.