18.306 Advanced Partial Differential Equations with Applications
Fall 2009

Answers to Problem Set Number 01 for 18.306 — MIT (Fall 2009)

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1 Linear 1st order PDE (problem 08).

1.1 Statement: Linear 1st order PDE (problem 08).

Solve the problem below, using the method of characteristics: (a) Compute the characteristics, as done in the lectures, starting from each point in the data set. (b) Next solve for the solution u along each characteristic. (c) Finally, eliminate the characteristic variables ζ and s from the expression[1] for u obtained in step (b) — using the result in step (a) — to obtain the solution as a function of x and y.

$$(x-y)\,u_x + (x+y)\,u_y = x^2 + y^2, \tag{1.1}$$

with data $u(x, 0) = (1/2)\,x^2$ for $1 \leq x < \exp(2\pi)$.

Answer this question: *Where in the* $(x,\, y)$ *plane does the problem above define the solution* u*?* That is: *what is the region of the plane characterized by the property that: through each point in this region there is exactly one characteristic connecting it with the curve where the data is given?*

1.2 Answer: Linear 1st order PDE (problem 08).

The characteristic form of equation (1.1) is

$$\frac{dx}{ds} = x - y, \quad \frac{dy}{ds} = x + y, \quad \text{and} \quad \frac{du}{ds} = x^2 + y^2. \tag{1.2}$$

The given data then translates into the conditions

$$x = \zeta, \quad y = 0, \quad \text{and} \quad u = \frac{1}{2}\,\zeta^2, \tag{1.3}$$

for $1 \leq \zeta < e^{2\pi}$ and $s = 0$.

The equations in (1.2 — 1.3) have the solution

$$x = \zeta\, e^s \cos s, \quad y = \zeta\, e^s \sin s, \quad \text{and} \quad u = \frac{1}{2}\,\zeta^2\, e^{2\,s}. \tag{1.4}$$

[1]Here ζ is the label for each characteristic, and s is a parameter along the characteristics.

Clearly, for any fixed ζ, the curves in the plane that these formulas define are counter-clockwise logarithmic spirals, scaling by a factor $e^{2\pi}$ per cycle. Since $1 \le \zeta < e^{2\pi}$, these curves cover the whole plane (except the origin), with exactly one curve going through each point such that $x^2 + y^2 > 0$.

From (1.4), it follows easily that the solution to (1.1) is

$$u = \frac{1}{2}\left(x^2 + y^2\right), \tag{1.5}$$

which is defined (and unique) on $0 < x^2 + y^2 < \infty$.

Remark 1.1 *Notice that (1.5) has a unique limit as $x^2 + y^2 \to 0$. However, this is not generic. For example, consider the case when the data is $u(x,\, 0) = 0$ for $1 \le x < e^{2\pi}$. Then $u = \frac{1}{2}\zeta^2\left(e^{2s} - 1\right)$. But $x^2 + y^2 \to 0$ corresponds to $s \to -\infty$ along characteristics. Hence u has no limit as $x^2 + y^2 \to 0$.*

2 Well and/or ill posed PDE (problem 01).

2.1 Statement: Well and/or ill posed PDE (problem 01).

Let $\rho = \rho(x,\, t)$ be the density of some conserved quantity, and let $q = q(x,\, t)$ be the corresponding flux. Then, in the absence of sources:

$$\rho_t + q_x = 0. \tag{2.1}$$

Assume now that, while examining the physical problem leading to this equation, you convince yourself that a "good approximation" for the flux is

$$q = \rho + c\,\rho_x, \tag{2.2}$$

where c is some constant. Substituting (2.2) into (2.1) yields a pde for ρ. *What restriction should you impose on the constant c so that this pde is not ill-posed?* Specifically: what restriction on c guarantees that the equation does not exhibit arbitrarily large growth factors at high frequencies?

2.2 Answer: Well and/or ill posed PDE (problem 01).

The pde for ρ resulting from (2.1 – 2.2) is

$$\rho_t + \rho_x + c\,\rho_{xx} = 0. \tag{2.3}$$

It is easy to check that $\rho = \exp(i\,k\,x + \lambda\,t)$ is a solution of this equation if and only if

$$\lambda = -i\,k + c\,k^2. \tag{2.4}$$

Hence, in order to avoid unbounded growth as $|k| \to \infty$, we need to require $c \leq 0$.

Remark 2.1 *Equations such as (2.1) occur when some conserved quantity diffuses within a fluid. For example, let $\rho = \rho(\vec{x}, t)$ be the salt concentration in water moving at speed $\vec{u} = \vec{u}(\vec{x}, t)$. Now: (1) Salt diffuses from regions of higher concentration to those of lower concentration at a rate proportional to the gradient of the concentration, and along the gradient (this is Fick's Law). (2) Salt is carried by the fluid. Hence the salt flux has the form*

$$\vec{q} = \vec{u}\,\rho - \nu\,\mathrm{grad}\,\rho, \tag{2.5}$$

where the diffusion coefficient is positive: $\nu > 0$. This leads to the equation

$$\rho_t + \mathrm{div}(\vec{u}\,\rho) = \mathrm{div}(\nu\,\mathrm{grad}\,\rho). \tag{2.6}$$

In 1-D, and for u and ν constants, this has the form in (2.3), with $c < 0$.

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3 Semi-Linear 1st order PDE (problem 03).

3.1 Statement: Semi-Linear 1st order PDE (problem 03).

Consider the problem

$$u_t + u_x = u^2, \quad \text{for } \ t > 0 \quad \text{and} \quad -\infty < x < \infty, \tag{3.1}$$

with initial condition $\boldsymbol{u(x,\,0) = F(x) = 1/(1 - x + x^2)}$ — note that the denominator here is positive for all $-\infty < x < \infty$.

Where is the solution defined? Specifically: **(a)** Compute, explicitly, the boundary of the region where the solution is defined — t as a function of x, or x as a function of t. **(b)** Do a plot of the region where the solution is defined.

3.2 Answer: Semi-Linear 1st order PDE (problem 03).

The characteristic form of (3.1) is
$$\frac{du}{dt} = u^2 \quad \text{along} \quad \frac{dx}{dt} = 1. \tag{3.2}$$
The given data then translates into the conditions
$$x = \zeta \quad \text{and} \quad u = F(\zeta) = \frac{1}{1-\zeta+\zeta^2} \quad \text{at} \quad t = 0, \quad \text{for} \quad -\infty < \zeta < \infty. \tag{3.3}$$
It follows that
$$x = \zeta + t, \quad \text{and} \quad u = \frac{F(\zeta)}{1 - t\,F(\zeta)}. \tag{3.4}$$
Clearly, since $F > 0$ everywhere, along each characteristic the solution is defined for $0 \le t < 1/F(\zeta)$ only, and becomes singular ($u \to \infty$) at $t = 1/F(\zeta)$. Hence *the boundary of the region where the solution is defined is given by the curve* (**singularity boundary**) described parametrically by
$$t = \frac{1}{F(\zeta)} = 1 - \zeta + \zeta^2 \quad \text{and} \quad x = \zeta + \frac{1}{F(\zeta)} = 1 + \zeta^2, \tag{3.5}$$
where $-\infty < \zeta < \infty$. Thus $\zeta = \pm\sqrt{x-1}$, where $\boldsymbol{x \ge 1}$, and
$$t = x \pm \sqrt{x-1}. \quad \text{Equivalently:} \quad (x-t)^2 = x - 1. \tag{3.6}$$
In terms of $X = x - t$ and $Y = x + t$, this has the form
$$Y = 2\left(X - \frac{1}{4}\right)^2 + \frac{15}{8}. \tag{3.7}$$
Hence, the singularity boundary is a parabola — see figure 3.1

Remark 3.1 *Notice that, from (3.4), we can write the solution (explicitly) as a function of x and t. Namely*
$$u = \frac{F(x-t)}{1 - t\,F(x-t)} = \frac{1}{1 - x + (x-t)^2}. \tag{3.8}$$
This yields a formula for the solution even inside the shaded region in figure 3.1, but **the data uniquely defines the solution for $t \ge 0$, and (x, t) outside the shaded region ONLY.** *For example:*
$$u = \begin{cases} \dfrac{1}{1 - x + (x-t)^2} & \text{for} \quad 1 - x + (x-t)^2 > 0, \\ 0 & \text{for} \quad 1 - x + (x-t)^2 < 0, \end{cases} \tag{3.9}$$
also defines a solution, satisfying the data, and (also) singular on the curve $1 - x + (x-t)^2 = 0$. Which one is the "correct one"? Without additional information, we cannot tell.

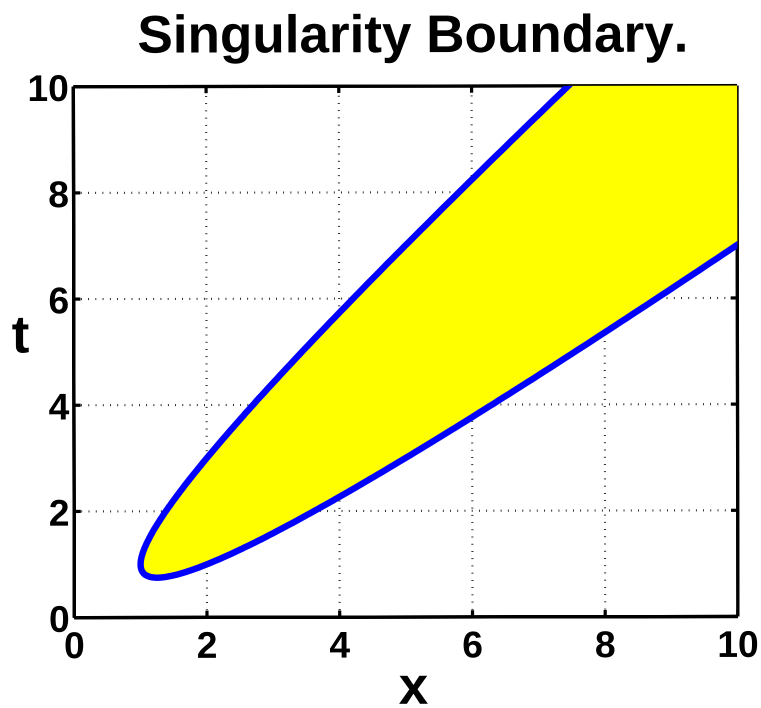

Figure 3.1: Singularity boundary for the solution of the problem in (3.1). The solution is defined for $t > 0$, outside the shaded region ONLY — that is, for $\boldsymbol{1 - x + (x - t)^2 > 0}$.

4 Quasi-Linear 1st order PDE (problem 01).

4.1 Statement: Quasi-Linear 1st order PDE (problem 01).

Consider the problem $$u_t + u\, u_x = 0, \quad \text{for } t > 0 \quad \text{and} \quad -\infty < x < \infty, \tag{4.1}$$

with initial condition $\boldsymbol{u(x,\, 0) = F(x) = -\arctan(x)}$ for $-\infty < x < \infty$.

Where is the solution defined?

- Compute the characteristics, and find the region in $t > 0$ characterized by the property: exactly one characteristic goes trough each point in it. Notice that if $x = X(\zeta,\, t)$ is the formula for the characteristics — with ζ given by $X(\zeta,\, 0) = \zeta$ — then, for any fixed time t, the multiple valued region is in-between the points where X_ζ vanishes (justify this; see (a – c) below).
- Plot of the region in space-time where the characteristics give a multiple valued answer.

Note that $F = F(x)$ above has the following properties

(a) $G = dF/dx < 0$. **(b)** $G(x)$ vanishes as $x \to \pm\infty$. **(c)** G has a single minimum.

4.2 Answer: Quasi-Linear 1st order PDE (problem 01).

The characteristic form of (4.1) is $\dfrac{du}{dt} = 0 \quad \text{along} \quad \dfrac{dx}{dt} = u.$ (4.2)

The given data translates into the conditions

$$x = \zeta \quad \text{and} \quad u = F(\zeta) = -\arctan(\zeta) \quad \text{at} \quad t = 0, \quad \text{for} \quad -\infty < \zeta < \infty. \tag{4.3}$$

It follows that
$$x = \zeta + t\,F(\zeta) = X(\zeta,\,t), \quad \text{and} \quad u = F(\zeta). \tag{4.4}$$

This defines u in the space-time region Ω in $t > 0$ defined by: at each point in Ω there is exactly one characteristic going through it. Hence: *For any fixed $t > 0$, $(x,\,t) \in \Omega$ if and only if the equation $x = X(\zeta,\,t)$ has exactly one solution ζ.* In order to compute Ω, consider the partial derivative

$$X_\zeta = 1 + t\,G(\zeta), \quad \text{where} \quad G(\zeta) = \frac{dF}{d\zeta}(\zeta). \tag{4.5}$$

Now

1. From **(b)** it should be clear that $X_\zeta \to 1$ as $\zeta \to \pm\infty$.
2. From **(a)** it follows that: for t large enough, there will be values of ζ where X_ζ is negative.

Adding to this the information in **(c)** we conclude that, if we define

$$t_c = -\frac{1}{\min G(\zeta)} > 0, \tag{4.6}$$

then:

3. For $t < t_c$, $X_\zeta > 0$ everywhere.
4. For $t > t_c$, $X_\zeta < 0$ in some interval $\zeta_L < \zeta < \zeta_R$, and $X_\zeta > 0$ elsewhere.

We conclude that:

— X is an increasing function of ζ for $\zeta < \zeta_L$.

— X has a local maximum at $\zeta = \zeta_L$.

— X is a decreasing function for $\zeta_L < \zeta < \zeta_R$.

— X has a local minimum at $\zeta = \zeta_R$.

— X is an increasing function for $\zeta > \zeta_R$.

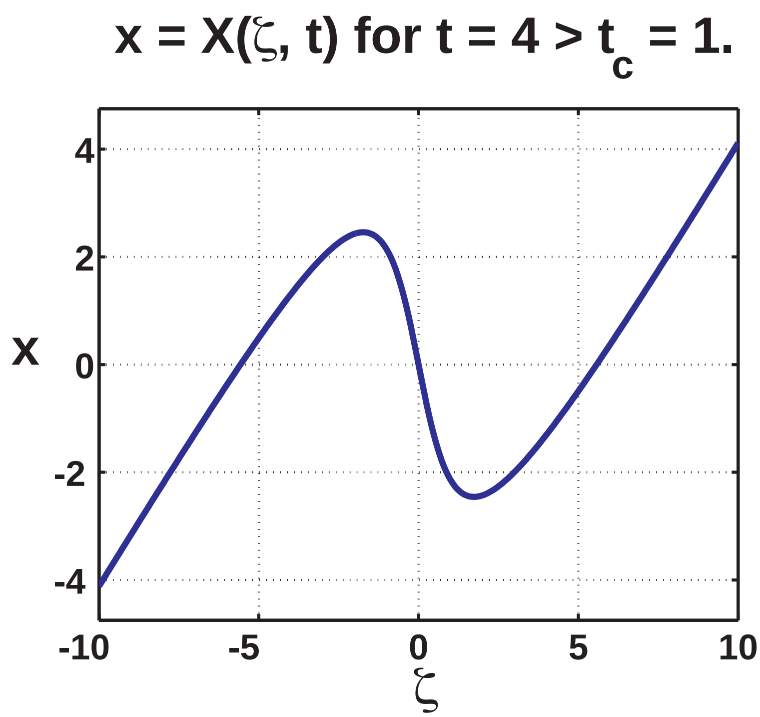

Figure 4.1: Plot of the characteristic coordinate $x = X(\zeta,\, t)$, as a function of ζ, for some fixed time $t > t_c$. Clearly: ζ, as a function of x, is multiple valued for x between the local minimum and the local maximum of X. Hence, for these values, at this given time, the solution to the problem in (4.1) cannot be defined in the absence of extra information — not provided by the problem statement.

— $X \to \pm\infty$ as $\zeta \to \pm\infty$.

Figure 4.1 shows a plot of $x = X$, as a function of ζ, for a fixed $t > t_c$.

From the above, it follows that **the region where the characteristics give multiple values for u** — hence, fail to define it — is given by[2]

$$x_R = X(\zeta_R, t) \le x \le x_L = X(\zeta_L, t) \quad \text{for} \quad t \ge t_c. \tag{4.7}$$

Of course, ζ_L and ζ_R follow easily from (4.5); they are the two solutions of the equation

$$G(\zeta) = -1/t, \quad \text{for} \quad t > t_c. \quad [\text{Note that } \zeta_L = \zeta_R \text{ for } t = t_c]. \tag{4.8}$$

[2]Note that X is decreasing between ζ_L and ζ_R, hence $x_R < x_L$.

Remark 4.1 *That (4.8) has two solutions for each $t > t_c$ follows from (a-c). From the definition of t_c and (b), it must have at least two. But more than two lead to a violation of (c).*

It follows that **the boundary of the multiple values region can be described parametrically by**

$$t = -1/G(\zeta) = 1 + \zeta^2 \quad \text{and} \quad x = \zeta - t\,\arctan(\zeta), \tag{4.9}$$

where we have used that, for $F(\zeta) = -\arctan(\zeta)$, $G(\zeta) = F'(\zeta) = -1/(1+\zeta^2)$. Alternatively:

$$\zeta_L = -\sqrt{t-1}, \quad \zeta_R = \sqrt{t-1}, \tag{4.10}$$

for $\boldsymbol{t > t_c = 1}$, with the multiple values occurring for (see figure 4.2)

$$\sqrt{t-1} - t\,\arctan(\sqrt{t-1}) \le x \le -\sqrt{t-1} + t\,\arctan(\sqrt{t-1}). \tag{4.11}$$

5 Quasi-Linear 1st order PDE (problem 02).

5.1 Statement: Quasi-Linear 1st order PDE (problem 02).

Consider the problem
$$u_t + u\,u_x = 0, \quad \text{for } t > 0 \quad \text{and} \quad -\infty < x < \infty, \tag{5.1}$$

with initial condition $\boldsymbol{u(x,\,0) = F(x)}$ for $-\infty < x < \infty$. Assume that the boundary of the region of multiple values for the solution by characteristics is known, and given by the equation

$$t = 1 + x^2, \quad \text{for } -\infty < x < \infty, \tag{5.2}$$

with multiple values in the region $t > 1 + x^2$. **QUESTION:** *What can you say about F? Can you determine it from the information given?* **NOTE: this problem is a little tricky!**

Remark 5.1 *This sounds as determining the past from the future: if F could be found, causality would be violated. However, this is not quite so. In cases where multiple values appear, the physically relevant solution does not allow access to the multiple values region's boundary. Usually one can get from it just a few points along this boundary. Hence, by giving access to the full curve, this problem is giving a lot of extra information that cannot be extracted from the physically relevant solution.*

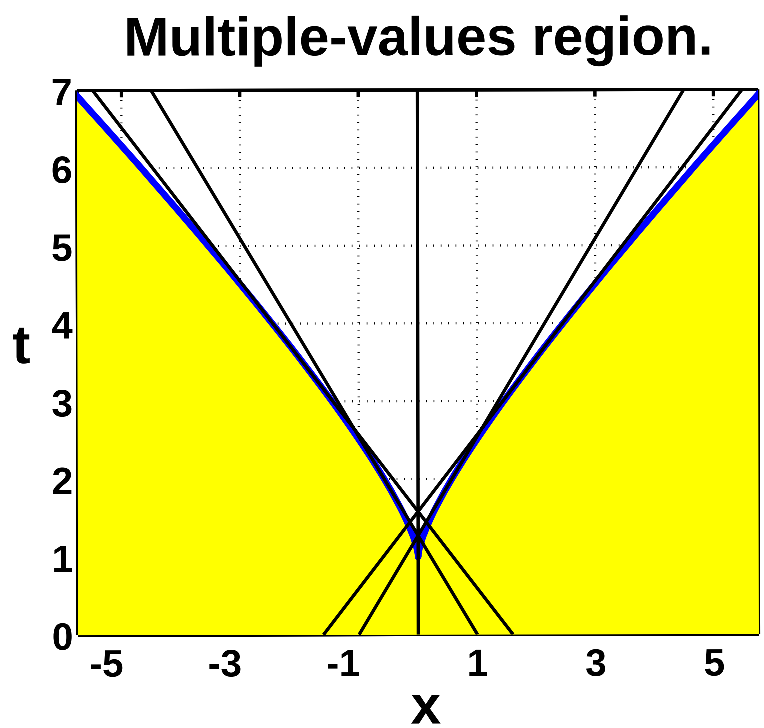

Figure 4.2: Multiple valued region for the solution by characteristics to the problem in (4.1). This is the V shaped region comprised between the two curves meeting at a cusp at $(x,\,t) = (0,\,0)$ — such cusps are typical for multiple valued regions arising from smooth initial data. A few characteristics have been drawn in the picture. Notice that the characteristics are tangent to the boundary of the multiple valued region — this is also typical.

5.2 Answer: Quasi-Linear 1st order PDE (problem 02).

The solution by characteristics of the problem in (5.1) is given by

$$u = F(\zeta) \quad \text{along the lines} \quad x = \zeta + t\,F(\zeta) = X(\zeta,\,t), \quad \text{where} \quad -\infty < \zeta < \infty. \tag{5.3}$$

Clearly, for each time $t > 0$, the region of multiple values corresponds to those values of x for which the equation for ζ given by $x = X(\zeta, t)$ has more than one solution $\zeta = Z(x, t)$.

Proof: Let $(x,\,t)$ be such that $\zeta_1 \neq \zeta_2$ exist satisfying $x = X(\zeta, t)$. Then $\zeta_1 - \zeta_2 = t\,(u_2 - u_1)$, where $u_j = F(\zeta_j)$. Since $\zeta_1 - \zeta_2 \neq 0$, $u_1 - u_2 \neq 0$.

Hence:

1 If F is continuous, multiple values occur (at any particular time) if and only if X fails to be monotone as a function of ζ. Namely: X must either have local maximums or minimums.

Then, in the equation $x = X(\zeta,\, t)$, as x goes down (resp. up) through a value corresponding to a local maximum (resp. minimum), two new solutions (at least) appear.

2 If F is differentiable, a necessary condition for multiple values to occur (at any particular t) is that $X_\zeta = 0$ somewhere. If $X_{\zeta\zeta} \neq 0$ at any of those points, then multiple values occur.

From **1** and **2** we arrive at the following result:

Lemma 5.1 *Let* Γ be a curve which is a part of the boundary of the region of multiple values *(arising from a differentiable F), which can be described by an equation of the form $x = f(t)$. Then the points in Γ must correspond to values of ζ where $X_\zeta = 0$. In other words,* **Γ must be a part of the curve (parameterized by ζ) described by the equations**

$$t = -1/G(\zeta) = \tau(\zeta) \quad \textit{and} \quad x = \zeta - F(\zeta)/G(\zeta) = \chi(\zeta), \quad -\infty < \zeta < \infty, \tag{5.4}$$

where $G = dF/d\zeta$.

Proof: $0 = X_\zeta = 1 + t\,G$ leads to the first equation. The second arises from substituting the value of t thus obtained into the equation for X.

Note: *the restriction that, along Γ, x should be a function of t follows because* **1-2** *apply at any fixed t, as x is varied and the number of values for the solution changes. If x is kept fixed, and t is varied, situations can arise where a change in the number of values cannot be associated with the vanishing of X_ζ — we will show an example later in this answer.*

From lemma **5.1** it follows that, if the boundary of the region of multiple values is known, then (at least in principle) it should be possible to find F, since then (5.4) can be used to obtain an ode for F. In particular, (5.2) leads to[3]

$$0 = 1 + \frac{1}{G} + \left(\zeta - \frac{F}{G}\right)^2, \tag{5.5}$$

which is a nonlinear, first order, ode for F, since $G = F'$. Hence, the problem has been reduced to that of solving an ode. However, this is easier said than done:

1. Equation (5.5) may be a first order scalar ode, but it is also non-constant coefficients, and with a nasty nonlinearity: Write the equation in the standard form $G = f(F,\, \zeta)$. Then f is multiple valued (it involves a square root), and yields imaginary values unless F and ζ are appropriately restricted.

[3]Note that (5.2) yields two curves satisfying the hypothesis of lemma **5.1**. Namely: $x = \pm\sqrt{t-1}$, with $t \geq 1$.

Hence, solving this equation is anything but trivial.

2. The general solution to (5.5) involves a free constant. Does this mean that there is a one parameter family of possible choices for F, all of them leading to the same region of multiple-values? Or is there some reason that allows only a single (or a few) choices for the parameter?

Rather than attempt to solve (5.5) directly, we will use here an alternative path, which leads to easily solvable equations, and no free parameters.

Using the definition for τ and χ in (5.4), $G = dF/d\zeta$, and $\tau = 1 + \chi^2$, we can write

$$\frac{d\chi}{d\zeta} = F\,\frac{d\tau}{d\zeta} = 2F\chi\,\frac{d\chi}{d\zeta}. \tag{5.6}$$

Hence, either $d\chi/d\zeta = 0$ or $2F\chi = 1$. We analyze these two cases below.

The case $d\chi/d\zeta = 0$.

Clearly, if $\chi(\zeta) \equiv x_0$ is constant, then the *boundary of the region of multiple values is just a point, not a parabola!* Hence, this case is not relevant to the problem at hand. Nevertheless, let us continue the analysis, since having the boundary of the region of multiple values as a single point is a rather puzzling statement; **What does this mean?**

Substituting $\chi \equiv x_0$ into the definition for χ in (5.4) yields

$$\frac{G}{F} = \frac{1}{\zeta - x_0} \quad \Longrightarrow F = g\,(\zeta - x_0) \quad \text{and} \quad G = g = -\frac{1}{\tau}, \tag{5.7}$$

where g is some constant. But $\tau = 1 + \chi^2 = 1 + x_0^2$, so we end up with

$$F(\zeta) = -\frac{\zeta - x_0}{1 + x_0^2}. \tag{5.8}$$

For this initial data, it is easy to eliminate ζ from (5.3), and obtain the following explicit expression for the solution to (5.1)

$$u = \frac{x - x_0}{t - 1 - x_0^2}. \tag{5.9}$$

This is a straight line with slope $1/(t - 1 - x_0^2)$. At $t = 1 + x_0^2$, the line becomes vertical at $x = x_0$, and the whole solution ceases to exist. *There is no "region of multiple values" to speak of* — though, right at $x = x_0$ and $t = 1 + x_0^2$, u takes all possible values.

The case $2F\chi = 1$.

Substituting $\chi = 1/(2\,F)$ and $\tau = 1 + \chi^2 = 1 + 1/(4\,F^2)$ into the definitions for τ and χ in (5.4) yields

$$0 = \left(1 + \frac{1}{4\,F^2}\right) G + 1 \quad \text{and} \quad 0 = \left(\frac{1}{2\,F} - \zeta\right) G + F. \tag{5.10}$$

Using the equation on the left to eliminate G on the equation on the right yields the equivalent set

$$0 = \frac{d}{d\zeta}\left(F - \frac{1}{4\,F} + \zeta\right) \quad \text{and} \quad 0 = -\left(\frac{1}{2\,F} - \zeta\right) + F + \frac{1}{4\,F} = F - \frac{1}{4\,F} + \zeta. \tag{5.11}$$

This has the following two solutions

$$F = \frac{-\zeta \pm \sqrt{1+\zeta^2}}{2}. \tag{5.12}$$

It would seem that we are done now, with F as given by either of the two possibilities in (5.12) being the answer to this problem. But **this is not quite so:** all we know so far is that, if the region of multiple values is given by $t > 1 + x^2$, then F must be as in (5.12). But: is the region of multiple values that corresponds to (5.12) actually $t > 1 + x^2$? This we must still check. Thus the next task is to *calculate directly what the region of multiple values that (5.12) leads to is.*

Consider the case when (the analysis for other case is quite similar)

$$F = \frac{-\zeta + \sqrt{1+\zeta^2}}{2}. \tag{5.13}$$

Then $\boldsymbol{F \sim \dfrac{1}{4\,\zeta}}$ as $\zeta \to \infty$, $\boldsymbol{F \sim -\zeta - \dfrac{1}{4\,\zeta}}$ as $\zeta \to -\infty$,

$$\frac{dF}{d\zeta} = -\frac{1}{2} + \frac{\zeta}{2\sqrt{1+\zeta^2}} < 0, \quad \text{and} \quad \frac{d^2F}{d\zeta^2} = \frac{1}{2\,(1+\zeta^2)^{3/2}} > 0. \tag{5.14}$$

Thus F is monotone decreasing, and $G = F'$ is monotone increasing — from $G(-\infty) = -1$ to $G(\infty) = 0$. It is then easy to see that:

A. For each $0 \le t < 1$, $x = X(\zeta, t)$ is a monotone increasing function of ζ, going from $X = -\infty$ at $\zeta = -\infty$ to $X = \infty$ at $\zeta = \infty$. Hence, for this time range, the solution u is single valued, and defined for all values of x. Furthermore, since F is monotone decreasing, going from ∞ on the left to 0 on the right, so is u. All of this is illustrated by the plots in figure 5.1.

Proof of **A**: Since $0 \le t < 1$, and $G > -1$, $X_\zeta = 1 + G(\zeta)\,t > 0$. Furthermore: $X \sim \zeta$ as $\zeta \to \infty$, and $X \sim (1-t)\,\zeta - \dfrac{t}{4\,\zeta}$ as $\zeta \to \infty$.

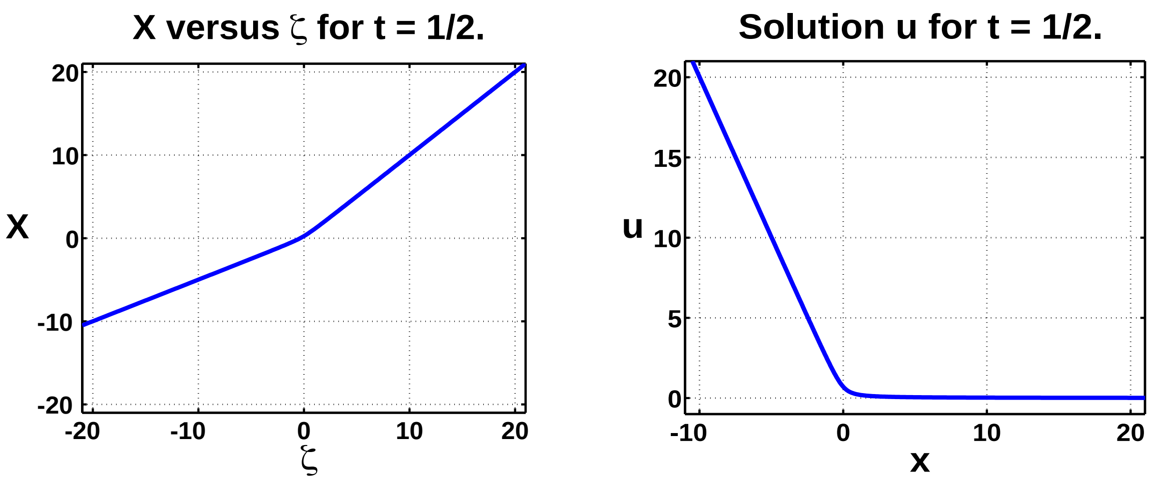

Figure 5.1: Solution by characteristics of the problem in (5.1), for F as in (5.13). Left panel: plot of the characteristic coordinate $x = X(\zeta,\, t)$ — defined in (5.3) — as a function of ζ, for some fixed time $0 \leq t < t_c = 1$. Right panel: plot of the solution u, as a function of x, for the same time t.

B. For $t = t_c = 1$, $x = X(\zeta, t)$ is a monotone increasing function of ζ, going from $X = 0$ at $\zeta = -\infty$ to $X = \infty$ at $\zeta = \infty$. Hence, for this time, the solution u is single valued, but defined for $x > 0$ only. Again, u is a monotone decreasing function of x, going from $u = \infty$ at $x = 0$ to $u = 0$ at $x = \infty$. All of this is illustrated by the plots in figure 5.2.

Proof of **B**: similar to the proof of **A**.

C. For $t > t_c = 1$, $x = X(\zeta, t)$ has a single global minimum somewhere (where $x = x_m$, say), and goes to infinity on either side as $\zeta \to \pm\infty$. Hence, for this time range, the solution u is defined (but double valued) for $x > x_b$ only. Furthermore, backtracking the calculations that lead us to the form for F in (5.13), it is easy to see that $x_m = \sqrt{t-1}$. All of this is illustrated by the plots in figure 5.3.

Proof of **C**: similar to the proof of **A**.

We conclude that **the region of multiple values corresponding to (5.13) is NOT $t > 1 + x^2$, but the region described by:**[4]

$$t > 1 \quad \text{and} \quad x > \sqrt{t-1}. \tag{5.15}$$

[4]Note the straight part of the boundary: $t = 1$ and $x \geq 0$, which is not associated with $X_\zeta = 0$. In fact, it arises because of the behavior of F as $\zeta \to -\infty$.

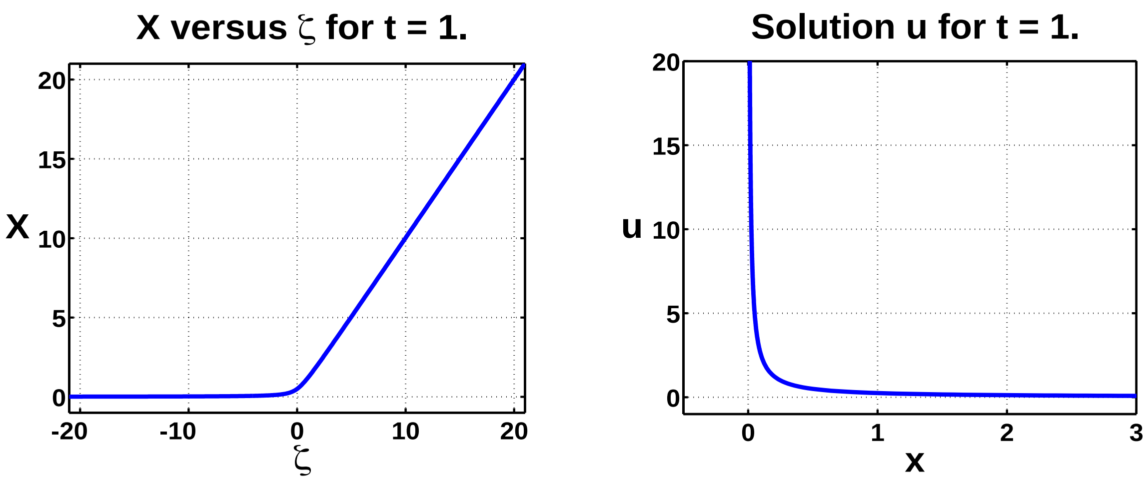

Figure 5.2: Solution by characteristics of the problem in (5.1), for F as in (5.13). Left panel: plot of the characteristic coordinate $x = X(\zeta,\, t)$ — defined in (5.3) — as a function of ζ, for time $t = t_c = 1$. Right panel: plot of the solution u, as a function of x, for the same time t.

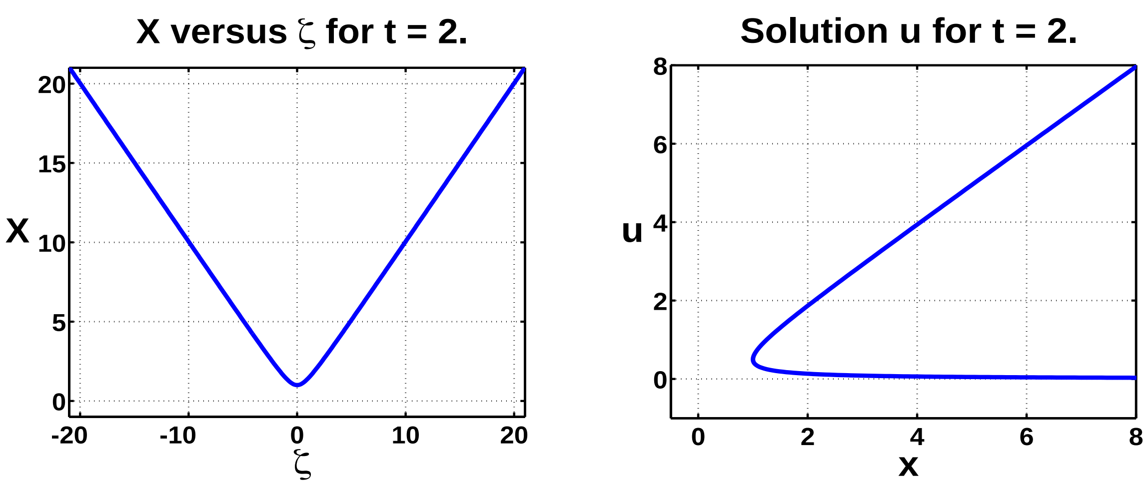

Figure 5.3: Solution by characteristics of the problem in (5.1), for F as in (5.13). Left panel: plot of the characteristic coordinate $x = X(\zeta,\, t)$ — defined in (5.3) — as a function of ζ, for some fixed time $t > t_c = 1$. Right panel: plot of the solution u, as a function of x, for the same time t.

Hence: we only **get 1/2 of the parabola, with the region of multiple values on the wrong side!**.

Remark 5.2 **Regarding the solutions to the nonlinear o.d.e. in (5.15).** *As a consequence of our "alternative" analysis, we now can write all the solutions to the nonlinear o.d.e. (5.15). These*

are given by the formulas in (5.8) and (5.12). Note that the solution in (5.8) has a parameter in it, while the solutions in (5.12) do not. This is because the two solutions in (5.12) are a special kind of solution, so called **envelope solutions** *— see below.*
Finally: consider the initial value problem for (5.5), where $F(0) = F_0$ is given. Then, for any F_0 such that $F_0^2 \neq 1/4$, the equation has two solutions[5] *— which have the form in (5.8). On the other hand, for $F_0 = \pm 1/2$, the solutions are unique, and given by (5.16).*

Envelope solutions: Consider a first order ode for $y = y(x)$

$$f(y,\, dy/dx, x) = 0, \tag{5.16}$$

where f is some smooth function. Assume that $\boldsymbol{y = Y(x,\ s)}$ is a one parameter family of solutions to this equation. Let $\boldsymbol{y = y_e(x)}$ be part of the envelope for the family $y = Y(x,\, s)$. Then:

The function $\boldsymbol{y = y_e(x)}$ solves (5.16), and it is called an **envelope solution.** (5.17)

Tasks left to the reader:

T1. Give a proof of (5.17).

T2. By direct substitution, show that (5.8) solves (5.5) for any x_0.

T3. Show that the envelope of the family of curves in (5.8), is given by the two curves in (5.12).

THE END.

[5] The reason for the two solutions is that (5.5) is, in fact, two equations — since it is quadratic in the derivative, with two possible values of G for almost all values of F (at $\zeta = 0$, the exceptions are $F = \pm 1/2$).