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18.306 Advanced Partial Differential Equations with Applications
Fall 2009

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Problem Set Number 02

18.306 — MIT (Fall 2009)

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October 10, 2009

Due: Friday October 16.

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1.1 Statement: Linear 1st order PDE (problem 09).

Surface Evolution. The evolution of a material surface can (sometimes) be modeled by a pde. In evaporation dynamics, where the material evaporates into the surrounding environment, consider a surface described in terms of its “height” $h = h(x, y, t)$ relative to the (x, y) -plane of reference. Under appropriate conditions, a rather complicated pde can be written¹ for h . Here we consider a (drastically) simplified version of the problem, where the governing equation is

$$h_t = \frac{A}{r} h_r, \quad \text{for } r = \sqrt{x^2 + y^2} > 0 \text{ and } t > 0, \quad \text{where } A > 0 \text{ is a constant.} \quad (1.1)$$

Axial symmetry is assumed, so that $h = h(r, t)$. Obviously, **h should be an even function of r** . This is both evident from the symmetry, and necessary in the equation to avoid singular behavior

¹From mass conservation, with the details of the physics going into modeling the flux and sink/source terms.

at the origin. Assume now

$$h(r, 0) = H(r^2), \tag{1.2}$$

where H is a smooth function describing a localized bump. Specifically: **(i)** $H(0) > 0$, **(ii)** H is monotone decreasing. **(iii)** $H \rightarrow 0$ as $r \rightarrow \infty$. **Note that** $h(r, 0)$ is an even function of r .

1. Using the theory of characteristics, write an explicit formula for the solution of (1.1 – 1.2).
2. Do a sketch of the characteristics in space time — i.e.: $r > 0$ and $t > 0$.
3. What happens with the characteristic starting at $r = \zeta > 0$ and $t = 0$ when $t = \zeta^2/\sqrt{2A}$?
4. Show that the resulting solution is an even function of r for all times.
5. Show that, as $t \rightarrow \infty$, the bump shrinks and vanishes.

1.2 Statement: Quasi-Linear 1st order PDE (problem 02) xtra Q's.

Do the "Tasks left to the reader" at the end of the answer to the last problem in problem set # 1 — i.e.: The answer to the *Quasi-Linear 1st order PDE (problem 02)*.

1.3 Statement: Linear 1st order PDE (problem 10).

Integrating factors. Show that the pde

$$(a(x, y) \mu)_y = (b(x, y) \mu)_x \tag{1.3}$$

is a necessary and sufficient condition guaranteeing that $\mu = \mu(x, y) \neq 0$ is an integrating factor for the ode

$$a(x, y) dx + b(x, y) dy = 0, \tag{1.4}$$

in any open subset of the plane without holes.

Part II. Assume that $a = 3xy + 2y^2$ and $b = 3xy + 2x^2$.

Find an integrating factor for (1.4) — i.e.: obtain a nontrivial solution of (1.3). **Use it to integrate** (1.4), and **write (1.4) in the form**

$$\Phi(x, y) = \text{constant}, \quad \text{for some function } \Phi. \tag{1.5}$$

Hint 1.1 Solving by characteristics (1.3) leads to (1.4), or equivalent, as part of the process — check this! To get out of this circular situation, note that: for a and b as above, $\mu = F(x, y)$ solves

(1.3) iff $\mu = F(y, x)$ does. This suggests that you should look for solutions² invariant under this symmetry; namely: $\mu(x, y) = \mu(y, x)$. Hence write $\boldsymbol{\mu} = \boldsymbol{\mu}(\mathbf{u}, \mathbf{v})$, with $\mathbf{u} = \mathbf{x} + \mathbf{y}$ and $\mathbf{v} = \mathbf{x} \mathbf{y}$, since solutions that satisfy $\mu(x, y) = \mu(y, x)$ must have this form — see remark 1.1.

Remark 1.1 The transformation $(x, y) \rightarrow (u, v)$ is not one to one: it maps the whole xy -plane into the region $v \leq \frac{1}{4}u^2$ of the uv -plane, with double valued inverse $x = \frac{1}{2}(u \pm \sqrt{u^2 - 4v})$ and $y = \frac{1}{2}(u \mp \sqrt{u^2 - 4v})$. Furthermore: **(a)** The two inverses are related by the $x \leftrightarrow y$ switch. **(b)** The singular line $u^2 = 4v$ corresponds to the line $x = y$. **(c)** The map is a bijection between the regions $x \leq y$ and $4v \leq u^2$. **(d)** The map is a bijection between the regions $x \geq y$ and $4v \leq u^2$. From **(c - d)** we see that: for any $\mu = \mu(x, y)$, $\mu = f(u, v)$ for $x \leq y$ and $\mu = g(u, v)$ for $x \geq y$, for some f and g . Then, if $\boldsymbol{\mu}(\mathbf{x}, \mathbf{y}) = \boldsymbol{\mu}(\mathbf{y}, \mathbf{x})$, $f = g$, so that $\boldsymbol{\mu}$ has the form $\boldsymbol{\mu} = \boldsymbol{\mu}(\mathbf{u}, \mathbf{v})$.

Part III. Why is it that this approach CANNOT be generalized to three variables? That is, to find integrating factors for equations of the form

$$a(x, y, z) dx + b(x, y, z) dy + c(x, y, z) dz = 0. \quad (1.6)$$

Note: this problem is based on Levine's problem 11 in chapter 9.

1.4 Statement: Quasi-Linear 1st order PDE (problem 03).

Simple waves in Gas Dynamics. Under the isentropic flow assumption, the Euler equations of Gas Dynamics in one dimension can be written in the form

$$\rho_t + (\rho u)_x = 0, \quad \text{conservation of mass,} \quad (1.7)$$

$$(\rho u)_t + (\rho u^2 + p)_x = 0, \quad \text{conservation of momentum,} \quad (1.8)$$

where ρ is the mass density, u is the flow velocity, and p is the pressure. In addition, an **equation of state** must be provided:

$$p = P(\rho), \quad \text{satisfying} \quad \frac{dP}{d\rho} = c^2 > 0, \quad (1.9)$$

where $\mathbf{c} = \mathbf{c}(\rho) > \mathbf{0}$ has the dimensions of a velocity (it is the sound speed).

²Note that you only need **one** nontrivial solution.

This is a system of two equations for two unknowns. Interestingly, the system has solutions that depend on a single unknown function. Namely, solutions of the form

$$\rho = R(\psi) \quad \text{and} \quad u = U(\psi), \quad (1.10)$$

where R and U are functions of the single argument ψ , and $\psi = \psi(x, t)$ satisfies a **scalar** quasi-linear equation of the form:

$$\psi_t + \lambda(\psi) \psi_x = 0. \quad (1.11)$$

Your **TASK** is to **find these solutions; that is: find R , U , and λ .**

Hints. (i) Characterize the functions R and U as solutions of an ode — do not seek explicit expressions. (ii) After you substitute (1.10) into the equations, cast the system into the form $AY = 0$, where A is a 2×2 matrix. Then nontrivial solutions will exist if and only if $\det(A) = 0$.

1.5 Statement: PDE numerical integration (problem 01).

Consider the linear pde

$$u_t + (c(x) u)_x = 0, \quad \text{where} \quad c = 1 + \frac{1}{4} \cos(x), \quad (1.12)$$

and **u is periodic of period 2π** — i.e.: $u(x + 2\pi, t) = u(x, t)$. Assume now that you are asked to calculate the solution of this pde for $0 \leq t \leq T = 6$, with **initial condition** given by

$$u(x, 0) = u_0(x) = \exp(-x^2) \quad \text{for} \quad -\pi \leq x \leq \pi, \quad (1.13)$$

extended periodically outside the interval $[-\pi, \pi]$.

You can extract a lot of information from the solution by characteristics of the problem above, but actual numerical values are not easy to access from it. For this, the best thing to do is to integrate the problem numerically. Here we will consider a few naive numerical algorithms for this purpose.

First, introduce a numerical grid, as follows:

$$x_n = -\pi + n h \quad \text{for} \quad 1 \leq n \leq N, \quad \text{and} \quad t_m = m k \quad \text{for} \quad 0 \leq m \leq M, \quad (1.14)$$

where N and M are “large” integers, $h = 2\pi/N$, and $k = T/M$. Let **u_n^m be the numerical solution’s grid point values**. The expectation is that these values will be related to the exact solution $u(x, t)$ by **$u_n^m = u(x_n, t_m) + \text{small error}$** , with the error vanishing as N and M grow.

Next, consider the following numerical discretizations of the problem, which arise upon replacing the derivatives in the equation by finite differences that approximate them up to errors of some positive order in h or k . **In all cases the formulas apply for $m \geq 0$, with $u_n^0 = u_0(x_n)$.**

$$\mathbf{A.} \quad 0 = \frac{1}{k} (u_n^{m+1} - u_n^m) + \frac{1}{h} (c(x_n) u_n^m - c(x_{n-1}) u_{n-1}^m).$$

$$\mathbf{B.} \quad 0 = \frac{1}{k} (u_n^{m+1} - u_n^m) + \frac{1}{h} (c(x_{n+1}) u_{n+1}^m - c(x_n) u_n^m).$$

$$\mathbf{C.} \quad 0 = \frac{1}{k} (u_n^{m+1} - u_n^m) + \frac{1}{2h} (c(x_{n+1}) u_{n+1}^m - c(x_{n-1}) u_{n-1}^m).$$

In all cases, once u_n^m is known for some m and all n , u_n^{m+1} can be explicitly computed, for all n . **Note:** when a formula above in **A**, **B** or **C**, calls for a value u_n^m outside the range $1 \leq n \leq N$, the periodic boundary conditions, which translate into $u_{n+N}^m = u_n^m$, must be used.

The **tasks in this problem are:**

- 1. Causality and numerics.** Using arguments based **solely** on how the information propagates in the exact solution (characteristics), versus how it propagates in the numerical schemes above, **argue that:** (1.1) *One of the schemes above cannot possibly work.* (1.2) *A necessary condition for the other two to work is that a restriction of the form $k \leq \lambda h$ be imposed on the time step — where $\lambda > 0$ is a constant that depends on $c = c(x)$.*
- 2.** Implement the schemes and try them out with various space resolutions,³ and a corresponding time resolution. Do you see convergence? Do the results agree with your analysis in item **1**? **Report what you see, and illustrate it with plots** — a few well selected plots should be enough!

THE END.

³ N in the range $20 \leq N \leq 200$ should be more than enough to see what happens.