

MIT OpenCourseWare
<http://ocw.mit.edu>

18.306 Advanced Partial Differential Equations with Applications
Fall 2009

For information about citing these materials or our Terms of Use, visit: <http://ocw.mit.edu/terms>.

Exam Number 01

18.306 — MIT (Fall 2009)

Rodolfo R. Rosales

MIT, Math. Dept., Cambridge, MA 02139

October 31, 2009

Due: Mon. Nov. 09.

Contents

1.1 Dispersive Waves and Modulation.	1
1.2 Discontinuous Coefficients in Linear 1st order p.d.e. #01.	1
1.3 P.D.E. blow up.	1
1.4 Statement: Initial Values for a Kinematic Wave (problem 03).	2

1.1 Dispersive Waves and Modulation.

This is a problem 3.2, starting in page 13 in the "problem set list."

1.2 Discontinuous Coefficients in Linear 1st order p.d.e. #01.

This is a problem 1.8, starting in page 6 in the "problem set list."

1.3 P.D.E. blow up.

This is a problem 12.3, starting in page 71 in the "problem set list."

1.4 Statement: Initial Values for a Kinematic Wave (problem 03).

Let u be the density of some conserved quantity, governed by the equation

$$u_t + \left(\frac{1}{2} u^2\right)_x = 0, \quad -\infty < x < \infty, \quad (1.1)$$

where *shocks should be used to avoid multiple values*. Consider now the following **initial value** $u(x, 0) = -\sin(x)$, and answer/perform the following questions/tasks:

- 1. When and where do shocks form?** Find the places in space-time at which a shock has to be started to prevent the characteristics from crossing, and stop multiple values from arising.
- 2. Where are the shocks located, for all times? Explicit formula required.** Find the shock path, for each shock identified in item **1**. Hint: Do not attempt to explicitly compute u on each side of the shock, to then solve the shock equations. Instead, show (e.g.: using the characteristic equations) that a certain symmetry applies to the solution. Then use it to get the shock position.
- 3. What does the solution look like as $t \rightarrow \infty$? Give a complete, and explicit, description** — valid with errors much smaller than $1/t$. Hint: The characteristics end when they encounter a shock. Thus, argue¹ that: as $t \rightarrow \infty$, only the characteristics starting near certain points remain in the solution. Use this to get an approximate description of the solution in this limit.
- 4. Extra credit:** Calculate the first correction to the approximation in item **1**.
- 5. Find** (a parametric description of) **the envelope of the characteristics for $t > 0$, and plot it**. Furthermore: **(5.1)** Note that the envelope has many branches, **describe them all**. **(5.2)** Describe the **asymptotic behavior of each branch** for $t \rightarrow \infty$.

Hint. *The initial value is periodic of period 2π . Argue² that this period should carry over to the characteristics — hence to the solution, shocks, envelope of the characteristics, etc. Thus you only have to describe one copy only of each of the objects that you are asked to analyze in this problem.*

THE END.

¹When I say argue, I really mean, argue! Using this just because I said so in the hint is not acceptable.

²When I say argue, I really mean, argue! Using this just because I said so in the hint is not acceptable.