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18.306 Advanced Partial Differential Equations with Applications
Fall 2009

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Problem Set Number 04

18.306 — MIT (Fall 2009)

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1.1 Statement: Eikonal equation (problem 01).

Consider the Eikonal equation (for the wave equation in 2-D) in a context where the wave speed is a constant (homogeneous media), so that we can set (upon non-dimensionalization) $c = 1$. Then

$$\phi_x^2 + \phi_y^2 = 1. \tag{1.1}$$

Consider now the situation where the wave-front $\phi = 0$ is a parabola. Specifically:

$$\phi = 0 \quad \text{on} \quad y = x^2, \tag{1.2}$$

with propagation direction towards y increasing. For this problem, **this is what you should do:**

1. Find the family of all the rays (characteristics) for $t > 0$. The easiest way is to describe it parametrically: $x = x(s, t)$ and $y = y(s, t)$, where $x(s, 0) = s$, $y(s, 0) = s^2$, and t is time of travel along the ray (for the wave-fronts) starting from the initial wave-front — i.e.: $\phi = t$.
2. Find the caustic. The caustic is the envelope of the family of rays = the locus of the intersections of infinitely close neighbors in the family of rays¹ = a curve such that each point in it belongs to one of the rays, and it is tangent to the ray there.² In this case of constant wave speed, where the rays are straight lines, the caustic is also the locus of the centers of curvature of the wave-fronts (all the wave-fronts have the same set of centers of curvature).

From the parametric description of the family of rays in item **1**, you should be able to obtain the caustic parametrically in terms of s . However, you should also be able to find a very simple formula — of the form $(y - y_a)^\alpha = \text{const.} (x - x_a)^2$ — for the caustic. Do so.

3. Do a sketch of the wave-front $\phi = 0$, and of the caustic. Indicate the region of the plane where the rays cross and give rise to multiple values in the solution to the equation.
4. The earliest time at which a ray crossing occurs corresponds to the singular point in the caustic (the arête). Find the position of the arête in space, the ray, and the time (or wave-front, as $\phi = t$) it corresponds to. Let these parameters be x_a , y_a , s_a , and t_a . Explicitly show that t_a is the earliest time at which a crossing of rays occurs.
5. Add to the sketch in item **3** the wave-front $\phi = t_a$. This wave-front is singular at the arête; describe the nature of this singularity. In particular, show that the wave-front satisfies (at leading order) a formula of the form $(y - y_a) \sim \text{const.} (x - x_a)^\mu$ near the arête.

¹ $(x_*, y_*) \in \text{caustic} \iff x_* = x(s, t) = x(s + ds, t + dt)$ and $y_* = y(s, t) = y(s + ds, t + dt)$, for some s and t .

² $(x_*, y_*) \in \text{caustic} \iff$ for some s and t : $x_* = x(s, t)$, $y_* = y(s, t)$, and $(x_t(s, t), y_t(s, t))$ is tangent to the caustic at (x_*, y_*) .

1.2 Statement: Eikonal equation (problem 03).

Consider the Eikonal equation (for the wave equation in 2-D) in a context where the wave speed is a constant (homogeneous media), so that one can set (upon non-dimensionalization) $c = 1$. Then

$$\phi_x^2 + \phi_y^2 = 1. \quad (1.3)$$

The *rays* for this equation are the lines defined by

$$\frac{dx}{dt} = \phi_x \quad \text{and} \quad \frac{dy}{dt} = \phi_y. \quad (1.4)$$

The characteristic form for (1.3) is then the o.d.e. system composed by the two equation in (1.4), plus the following three additional equations

$$\frac{d\phi_x}{dt} = 0, \quad \frac{d\phi_y}{dt} = 0, \quad \text{and} \quad \frac{d\phi}{dt} = 1. \quad (1.5)$$

Your tasks are:

1. Find an equation for the evolution along the rays of the Hessian of ϕ . Namely, the matrix

$$M = \begin{bmatrix} \phi_{xx} & \phi_{xy} \\ \phi_{yx} & \phi_{yy} \end{bmatrix} \quad (1.6)$$

2. Solve the equation for M derived in item **1**, and write a formula giving the front curvature $\kappa = \phi_{xx} + \phi_{yy}$ along each ray, as a function of t , and the front curvature κ_0 on the ray at the wave front corresponding to $t = 0$ — recall that the wave fronts are given by $\phi(x, y) = t$.

Note: the formula for κ involves t and κ_0 **only**.

Hints: **(i)** Consider the second order derivatives (i.e. ∂_{xx}^2 , ∂_{xy}^2 , and ∂_{yy}^2) of equation (1.3). Then use that, for any $f = f(x, y)$, its derivative along the rays is given by $\frac{d}{dt} f = \phi_x f_x + \phi_y f_y$. **(ii)** To solve the equation for M , proceed as follows: Let M_0 be the value of M at $t = 0$. Define $W = (1 + M_0 t) M$, and write the equation W satisfies. Using that $W = M_0$ at $t = 0$, you should now be able to solve this equation by inspection. **(iii)** Once you have solved the equation for M , use it to get the behavior of the eigenvalues of M — note that κ is the sum of the eigenvalues. **(iv)** Finally, inspect the gradient of equation (1.3). What does it tell you about the eigenvalues of M ?

1.3 Statement: Eikonal equation (problem 04).

Consider the Eikonal equation (for the wave equation in 2-D) in a context where the wave speed is a constant (homogeneous media), so that one can set (upon non-dimensionalization) $c = 1$. Then

$$\phi_x^2 + \phi_y^2 = 1. \quad (1.7)$$

The *rays* for this equation are the lines defined by $\frac{dx}{dt} = \phi_x$ and $\frac{dy}{dt} = \phi_y$. (1.8)

Equation (1.7) accepts solutions with singularities along the rays. Specifically, solutions for which:

1. ϕ and $\nabla \phi$ are both continuous.
2. The second derivatives³ of ϕ exist, but they fail to be continuous along some ray.

Exhibit a solution with these properties, valid in some region of the plane — say $x > 0$.

Hint: A simple set of solutions for (1.7) is given by $\phi = \phi_0 + \sqrt{(x - x_0)^2 + (y - y_0)^2}$, where ϕ_0 is a constant and $P = (x_0, y_0)$ is a point in the plane — these solutions correspond to waves radiating from a single point in space. You can construct an example by gluing solutions of this type.

THE END.

³Namely: $\phi_{x,x}$, $\phi_{x,y}$, $\phi_{y,x}$, and $\phi_{y,y}$.