18.306 Advanced Partial Differential Equations with Applications
Fall 2009

Answers to Problem Set Number 03 for 18.306 — MIT (Fall 2009)

Rodolfo R. Rosales (MIT, Math. Dept., Cambridge, MA 02139).

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1 Linear 1st order PDE (problem 10).

1.1 Statement: Linear 1st order P.D.E. (problem 10).

Integrating factors. **Show that** the p.d.e.

$$(a(x,\, y)\,\mu)_y = (b(x,\, y)\,\mu)_x \tag{1.1}$$

is a necessary and sufficient condition guaranteeing that $\mu = \mu(x,\, y) \neq 0$ is an integrating factor for the ode

$$a(x,\, y)\,dx + b(x,\, y)\,dy = 0, \tag{1.2}$$

in any open subset of the plane without holes.

Part II. Assume that $\boldsymbol{a = 3\,x\,y + 2\,y^2}$ and $\boldsymbol{b = 3\,x\,y + 2\,x^2}$**.**
Find an integrating factor for (1.2) — i.e.: obtain a nontrivial solution of (1.1). **Use it to integrate** (1.2), and **write (1.2) in the form**

$$\Phi(x,\, y) = \text{constant}, \quad \text{for some function } \Phi. \tag{1.3}$$

Hint 1.1 *Solving by characteristics (1.1) leads to (1.2), or equivalent, as part of the process — <u>check this</u>! To* get out of this circular situation, note that: *for a and b as above, $\mu = F(x,\, y)$ solves (1.1) iff $\mu = F(y,\, x)$ does. This suggests that you should look for solutions*[1] *invariant under this symmetry; namely: $\mu(x,\, y) = \mu(y,\, x)$. Hence* write $\boldsymbol{\mu = \mu(u,\, v)}$, with $\boldsymbol{u = x + y}$ and $\boldsymbol{v = x\,y}$, *since solutions that satisfy $\mu(x,\, y) = \mu(y,\, x)$ must have this form — see remark **1.1**.*

Remark 1.1 *The transformation $(x,\, y) \to (u,\, v)$ is not one to one: it maps the whole xy-plane into the region $v \leq \dfrac{1}{4}\,u^2$ of the uv-plane, with double valued inverse $x = \dfrac{1}{2}\left(u \pm \sqrt{u^2 - 4\,v}\right)$ and*

[1] Note that you only need **one** nontrivial solution.

$y = \frac{1}{2}\left(u \mp \sqrt{u^2 - 4\,v}\right)$. *Furthermore:* **(a)** *The two inverses are related by the* $x \leftrightarrow y$ *switch.* **(b)** *The singular line* $u^2 = 4\,v$ *corresponds to the line* $x = y$*.* **(c)** *The map is a bijection between the regions* $x \le y$ *and* $4\,v \le u^2$*.* **(d)** *The map is a bijection between the regions* $x \ge y$ *and* $4\,v \le u^2$*. From* **(c - d)** *we see that: for* any $\mu = \mu(x,\, y)$, $\mu = f(u,\, v)$ *for* $x \le y$ *and* $\mu = g(u,\, v)$ *for* $x \ge y$*, for some* f *and* g*. Then,* if $\boldsymbol{\mu(x,\, y) = \mu(y,\, x)}$, $f = g$, *so that* $\boldsymbol{\mu}$ has the form $\boldsymbol{\mu = \mu(u,\, v)}$.

Part III. Why is it that this approach CANNOT be generalized to three variables? That is, to find integrating factors for equations of the form

$$a(x,\, y,\, z)\, dx + b(x,\, y,\, z)\, dy + c(x,\, y,\, z)\, dz = 0. \tag{1.4}$$

Note: this problem is based on Levine's problem 11 in chapter 9.

1.2 Answer: Linear 1st order P.D.E. (problem 10).

If (1.1) is satisfied in an open subset without holes, then there exists a function $\Phi = \Phi(x,\, y)$ such that

$$\Phi_x = a\,\mu \quad \text{and} \quad \Phi_y = b\,\mu. \tag{1.5}$$

In fact, pick a fixed point in the set, say Q. Then define

$$\Phi(x,\, y) = \int_\Gamma \mu\, a\, dx + \mu\, b\, dy, \tag{1.6}$$

where Γ is any curve (in the set) connecting Q to $(x,\, y)$. The value of this integral does not depend on the curve Γ, as follows from Green's theorem and (1.1) — i.e.: $\oint_\Lambda \mu\, a\, dx + \mu\, b\, dy = 0$ for any closed curve Λ in the set.[2] Hence (1.6) does define a function — which (obviously) satisfies (1.5).

Given (1.5), equation (1.2) yields

$$0 = \mu\, a\, dx + \mu\, b\, dy = \Phi_x\, dx + \Phi_y\, dy = d\Phi \quad \Longleftrightarrow \quad \Phi = \text{constant}. \tag{1.7}$$

Of course, for (1.7) to be of any use, μ must be non-trivial: $\mu = 0$ always works, but it also leads to the useless function $\Phi(x,\, y) \equiv$ constant.

[2] **Here is where having a set without holes matters:** if there are holes, then $\oint_\Lambda \mu\, a\, dx + \mu\, b\, dy = 0$ is not guaranteed, and (1.6) cannot be used to define a function.

Vice-versa, if μ is an integrating factor, there is some function Φ such that

$$\mu\, a\, dx + \mu\, b\, dy = d\Phi = \Phi_x\, dx + \Phi_y\, dy. \tag{1.8}$$

Hence (1.5) applies, from which (1.1) follows.

Part II. With $a = 3\, x\, y + 2\, y^2$, and $b = 3\, x\, y + 2\, x^2$, equation (1.1) takes the form

$$a\, \mu_y - b\, \mu_x = (b_x - a_y)\, \mu = (x - y)\, \mu. \tag{1.9}$$

In terms of the coordinates $u = x + y$ and $v = x\, y$, a little bit of algebra reduces this equation to

$$(y - x)\, (2\, u\, \mu_u - v\, \mu_v + \mu) = 0. \tag{1.10}$$

That is, **for $x \neq y$ we have**

$$2\, u\, \mu_u - v\, \mu_v + \mu = 0. \tag{1.11}$$

We **only need one non-trivial solution for this equation to find an integrating factor.** Nevertheless, next we find **all** the solutions, using characteristics.

The characteristic equations for (1.11) can be written in the form

$$\frac{du}{2\, u} = -\frac{dv}{v} = -\frac{d\mu}{\mu} = ds, \tag{1.12}$$

where s is a parameter along each characteristic. From the first equality it follows that $u\, v^2 = \zeta$, where ζ is a constant on each characteristic — which we use as a label for the characteristic curve. From the second equality it follows that $\mu/v = f$, where f is also a constant along each characteristic — hence $f = f(\zeta)$ must be some function of the characteristic label. Thus the general solution to (1.11) has the form

$$\mu = v\, f(u\, v^2) = x\, y\, f\left((x + y)\, x^2\, y^2\right), \tag{1.13}$$

where f is some arbitrary function (with, at least, one derivative).

Remark 1.2 *The formula in (1.13) follows from solving (1.11), which is equivalent to (1.9) only for $x < y$, or $x > y$. Hence, at this stage, the only thing that we can say about the general solution to (1.9) is that it has the form*

$$\mu = x\, y\, f\left((x + y)\, x^2\, y^2\right) \quad \textit{for } x > y, \quad \textit{and} \quad \mu = x\, y\, g\left((x + y)\, x^2\, y^2\right) \quad \textit{for } x < y,$$

for some arbitrary function f and g. However, evaluating along $x = y$ yields

$$x^2\, f\left(2\, x^5\right) = x^2\, g\left(2\, x^5\right).$$

Thus, we conclude that:

All the solutions to (1.9) satisfy $\boldsymbol{\mu(x,\, y) = \mu(y,\, x)}$, and have the form in (1.13)

— *with the same function f for all x and y.* This is rather interesting, since (generally) the fact that a p.d.e. has a symmetry **does not** imply that all the solutions have it too. *For example: the heat equation $T_t = T_{x\,x}$ is invariant under $x \leftrightarrow -x$, but $T = 2 + \sin(x)\, e^{-t}$ is a solution that is not invariant under $x \leftrightarrow -x$.*

We now take the simplest of the solutions in (1.13), $\mu = v = x\, y$, and plug it into equation (1.5). This yields

$$\Phi_x = a\, x\, y = 3\, x^2\, y^2 + 2\, x\, y^3 \quad \text{and} \quad \Phi_y = b\, x\, y = 3\, x^2\, y^2 + 2\, x^3\, y. \tag{1.14}$$

Thus $\boldsymbol{\Phi = x^3\, y^2 + x^2\, y^3 + \Phi_0}$ — where Φ_0 is a constant. Hence, from (1.7), it follows that: **For the case $\boldsymbol{a = 3\, x\, y + 2\, y^2}$ and $\boldsymbol{b = 3\, x\, y + 2\, x^2}$, (1.2) can be integrated to**

$$x^3\, y^2 + x^2\, y^3 = u\, v^2 = \text{constant}. \tag{1.15}$$

Part III. If μ is a non-trivial integrating factor for (1.4), then

$$\mu\, a\, dx + \mu\, b\, dy + \mu\, c\, dz = d\Phi, \tag{1.16}$$

for some function $\Phi = \Phi(x,\, y,\, z)$. This is equivalent to

$$\mu\, a = \Phi_x, \quad \mu\, b = \Phi_y, \quad \text{and} \quad \mu\, c = \Phi_z. \tag{1.17}$$

However, this then implies the equations

$$(\mu\, a)_y = (\mu\, b)_x, \quad (\mu\, c)_x = (\mu\, a)_z, \quad \text{and} \quad (\mu\, b)_z = (\mu\, c)_y. \tag{1.18}$$

Thus, we get three equations for a single unknown μ. This is an over-determined system that (generally) has only one solution: $\mu = 0$. In fact, in (1.18), multiply the first equation by c, the second equation by b, the third equation by a, add, and use the fact that we want $\mu \neq 0$. This then yields

$$0 = a\,(b_z - c_y) + b\,(c_x - a_z) + c\,(a_y - b_x). \tag{1.19}$$

Equivalently

$$0 = \mathbf{w} \cdot (\nabla \times \mathbf{w})\,, \quad \text{where} \quad \mathbf{w} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}. \tag{1.20}$$

Hence, if a, b, and c do not satisfy equation (1.20), (1.4) has no integrating factor.

TASK LEFT TO THE READER: *Show that, at least locally, (1.20) is sufficient to guarantee that (1.18) has a non-trivial solution.*

Hint 1.2 *If $\mathbf{w} \equiv 0$, then any μ solves (1.18). Hence, in any sufficiently small cube, we can assume that one of the components of $\mathbf{w}$ is never zero. Thus, without loss off generality, assume that $a \geq \delta > 0$ in the cube $-\epsilon < x,\, y,\, z < \epsilon$, where δ and $\epsilon > 0$ are constants. Then:* **I.** *Use the first equation in (1.18) to construct $\mu_0 = \mu_0(x,\, y)$ for $-\epsilon < x,\, y < \epsilon$.* **II.** *Use the second equation in (1.18), with $\mu(x,\, y,\, 0) = \mu_0(x,\, y)$, to define μ in a neighborhood in $\mathcal{R}^{\ni}$ of the square $-\epsilon < x,\, y < \epsilon$.*
III. *Show that the function μ that you just constructed solves (1.18). To do this:*
Define $\boldsymbol{\phi = (\mu\, a)_y - (\mu\, b)_x}$ and $\boldsymbol{\psi = (\mu\, c)_y - (\mu\, b)_z}$. Then: (i) Use (1.19) to find an algebraic relationship between ϕ and ψ. (ii) Use that μ satisfies the middle equation in (1.18), to derive a p.d.e. that ϕ satisfies. (iii) By construction $\phi = 0$ for $z = 0$. Use the p.d.e. in (ii) to conclude that $\phi \equiv 0$. (iv) Use (i) and (iii), conclude that $\psi \equiv 0$. Since μ satisfies the the middle equation in (1.18) by construction, this ends the proof.
Note: point out where the assumption $a > 0$ comes into play in your arguments.

Remark 1.3 *Of course, once a μ satisfying (1.18) is obtained, a Φ satisfying (1.17) is given by*

$$\Phi(x,\, y,\, z) = \int_\Gamma \mu\, a\, dx + \mu\, b\, dy, \tag{1.21}$$

where Γ is any curve connecting some fixed point Q with $(x,\, y,\, z)$. **Why is it that the integral in (1.21) depends ONLY on the endpoints of the curve Γ?**

ANSWER to the task left to the reader:

aT1. Construction of μ_0. Consider the first equation in (1.20), namely: $a\,\mu_y - b\,\mu_x = (b_x - a_y)\,\mu$, and restrict it to the plane $z = 0$. Since $a > 0$, the characteristics cross the x-axis. Thus, if μ is prescribed along the x-axis (say: $\mu(x,\,0,\,0) = 1$), a solution is (uniquely) determined in a neighborhood of the x-axis, as a function of $(x,\,y)$. Call this solution $\mu_0 = \mu_0(x,\,y)$.

aT2. Construction of μ. Consider the second equation in (1.20), namely: $a\,\mu_z - c\,\mu_x = (c_x - a_z)\,\mu$. Since $a > 0$, the characteristics cross the $z = 0$ plane. Thus, if we prescribe $\mu(x,\,y,\,0) = \mu_0(x,\,y)$, a solution is determined, uniquely, in a neighborhood of the $z = 0$ plane. We need to show now that this μ also satisfies the first and the third equations in (1.20) — namely, that: $\boldsymbol{\phi = (\mu\,a)_y - (\mu\,b)_x}$ and $\boldsymbol{\psi = (\mu\,c)_y - (\mu\,b)_z}$ both vanish.

aT3. ϕ and ψ are related by $a\,\psi = c\,\phi$. Proof: $a\,\psi = a\,((c_y - b_z)\,\mu + c\,\mu_y - b\,\mu_z) = b\,(c_x - a_z)\,\mu + c\,(a_y - b_x)\,\mu + a\,c\,\mu_y - a\,b\,\mu_z = b\,((\mu\,c)_x - (\mu\,a)_z) + c\,\phi = c\,\phi$, where we have used (1.19) for the second equality.

aT4. ϕ and ψ satisfy $\phi_z = \psi_x$. Proof: $\phi_z = (\mu\,a)_{z\,y} - (\mu\,b)_{z\,x} = (\mu\,c)_{x\,y} - (\mu\,b)_{z\,x} = \psi_x$, where we have used that, by construction, μ satisfies the second equation in (1.18).

aT5. From **aT3** and **aT4**, $\boldsymbol{\phi_z = \left(\frac{c}{a}\,\phi\right)_x}$. Since ϕ vanishes for $z = 0$, by the construction of μ, it follows that $\boldsymbol{\phi \equiv 0}$. Then **aT3** yields $\boldsymbol{\psi \equiv 0}$, since $a > 0$.

Q.E.D.

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2 Riemann Problems (problem 01).

2.1 Statement: Riemann Problems (problem 01).

Consider the following conservation law (in a-dimensional variables)

$$u_t + \left(\frac{1}{2}\,u^2\right)_x = 1, \quad \text{for} \quad -\infty < x < \infty \quad \text{and} \quad t > 0, \tag{2.1}$$

where u is conserved, and shocks are used to avoid multiple-valued solutions.

Find the solution to the Riemann problem for this equation. Namely, for the initial values

$$u(x,\,0) = a \quad \text{for} \;\; x < 0 \quad \text{and} \quad u(x,\,0) = b \quad \text{for} \;\; x > 0, \tag{2.2}$$

where a and b are arbitrary real constants $-\infty < a,\ b < \infty$. **Hint:** *The solution involves shocks, expansion fans, and regions where u depends on time only. Expansion fans are regions where all the characteristics emanate from a single point in space time.*

2.1.1 Justification of quadratic fluxes.

Here we justify the use of conservation laws of the form

$$u_t + \left(\frac{1}{2}\,u^2\right)_x = S, \quad \text{for} \quad -\infty < x < \infty \quad \text{and} \quad t > 0, \tag{2.3}$$

where S is some source term, u is conserved and can be both positive or negative, and shocks are used to avoid multiple-values in the solution.

Consider a scalar conservation law in 1-D, with a source term,

$$\rho_t + q_x = S, \tag{2.4}$$

where $\rho = \rho(x,\,t)$ is the density of some conserved quantity (hence $\rho \geq 0$), $q = Q(\rho)$ is the corresponding flux, and $S = S(\rho, x)$ is the density of sources/sinks. **Assume now that $\boldsymbol{S}$** is "small" — this is made precise below in item **2**. Then solutions where ρ is close to a constant should be possible. Hence let $\rho_0 > 0$ be some fixed (constant) density value, and proceed as follows:

1. Expand $\boldsymbol{Q}$ near $\boldsymbol{\rho_0}$ using Taylor's theorem

$$Q = q_0 + c_0\,(\rho - \rho_0) + \frac{s_1}{2\,\rho_0}\,(\rho - \rho_0)^2 + \ldots, \tag{2.5}$$

where q_0 is the flux for $\rho = \rho_0$, c_0 is the corresponding characteristic speed, and $\boldsymbol{s_1}$ is a constant with the dimensions of a velocity. We now **assume that $\boldsymbol{s_1 \neq 0}$; in fact, that**[3] $\boldsymbol{s_1 > 0}$. *Notice that s_1 is a measure of how nonlinear the equation in (2.4) is. The further away for zero s_1 is, the stronger the leading order nonlinear term in the equation is.*

2. Let $S_1 > 0$ be some "typical" value for the source term size, and let $L > 0$ be some "typical" length scale. Then **the source term is small in the sense that** $\boldsymbol{0 \ll \epsilon^2 = \dfrac{L\,S_1}{s_1\,\rho_0} \ll 1}$.

Introduce the a-dimensional variables

$$\tilde{x} = \frac{x - c_0\,t}{L} \ \text{ and } \ \tilde{t} = \frac{\epsilon\,s_1}{L}\,t, \quad \text{with} \quad \rho = \rho_0\,(1 + \epsilon\,u) \ \text{ and } \ S = S_1\,\tilde{S}. \tag{2.6}$$

Then (2.4) becomes

$$u_{\tilde{t}} + \left(\frac{1}{2}\,u^2 + O(\epsilon)\right)_{\tilde{x}} = \tilde{S}. \tag{2.7}$$

Upon neglecting the $O(\epsilon)$ term, this has the form in (2.3).

[3] If $\boldsymbol{s_1 < 0}$, a similar analysis is possible.

2.2 Answer: Riemann Problems (problem 01).

The characteristic form of the equation in (2.1) is .... $\boldsymbol{\frac{du}{dt} = 1}$ along $\boldsymbol{\frac{dx}{dt} = u.}$

This has the general solution .... $\boldsymbol{u = t + f(\zeta)}$ and $\boldsymbol{x = \frac{1}{2}t^2 + t\,f(\zeta) + \zeta.}$
where $\boldsymbol{f(\zeta) = u(\zeta,\, 0)}$ is given by the initial data.

In a **rarefaction fan** all the characteristics emanate from a single point (say $(x,\, t) = (0,\, 0)$), hence

$$u = t + f \quad \text{and} \quad x = \frac{1}{2}t^2 + t\,f, \quad \Longrightarrow \quad u = \frac{1}{2}t + \frac{x}{t}, \tag{2.8}$$

valid in the region $\boldsymbol{\frac{1}{2}t^2 + t\,f_1 \leq x \leq \frac{1}{2}t^2 + t\,f_2}$ — where $\boldsymbol{f_1 \leq f_2}$ are constants.

At shocks the **Rankine-Hugoniot jump condition,** and the **Entropy condition** reduce to

$$\frac{dx_s}{dt} = \frac{1}{2}(u_L + u_R) \quad \text{and} \quad u_L > u_R, \tag{2.9}$$

where u_L is the state immediately to the left of the shock, and u_R is the state immediately to the right of the shock.

Putting this all together, we arrive at the following solution to the Riemann problem.

Case $\boldsymbol{a \leq b}$.

$$u = \begin{cases} t + a & \text{for} \qquad\qquad\qquad x \;\leq\; \frac{1}{2}t^2 + a\,t, \\ \dfrac{1}{2}t + \dfrac{x}{t} & \text{for} \quad \frac{1}{2}t^2 + a\,t \;\leq\; x \;\leq\; \frac{1}{2}t^2 + b\,t, \\ t + b & \text{for} \quad \frac{1}{2}t^2 + a\,t \;\leq\; x. \end{cases} \tag{2.10}$$

Case $\boldsymbol{a > b}$.

$$u = \begin{cases} t + a & \text{for} \quad x \;<\; \frac{1}{2}t^2 + \frac{1}{2}(a+b)\,t, \\ t + b & \text{for} \quad x \;>\; \frac{1}{2}t^2 + \frac{1}{2}(a+b)\,t. \end{cases} \tag{2.11}$$

3 Riemann Problems (problem 02).

3.1 Statement: Riemann Problems (problem 02).

Consider the following conservation law (in a-dimensional variables) for the density u

$$u_t + \left(\frac{1}{2}u^2\right)_x = \delta(x), \quad \text{for} \quad -\infty < x < \infty \quad \text{and} \quad t > 0, \tag{3.1}$$

where shocks are used to avoid multiple-valued solutions, and $\delta(*)$ stands for Dirac's delta function.

Solve the Riemann problem for (3.1) given by the initial data

$$u(x,\,0) = a \quad \text{for} \;\; x < 0, \quad \text{and} \quad u(x,\,0) = b \quad \text{for} \;\; x > 0, \tag{3.2}$$

where $-\infty < a,\; b < \infty$ are arbitrary real constants. **Hint:** *The solution involves shocks, expansion fans,*[4] *and regions where u is constant. The information in § 3.1.1 should prove useful.*

3.1.1 What the equation means. Causality.

The delta function (point source term) has meaning via the integral form of the conservation law; namely:

$$\frac{d}{dt}\int_{x_L}^{x_R} u\,dx = 1 + \frac{1}{2}\,u^2(x_L,\,t) - \frac{1}{2}\,u^2(x_R,\,t), \tag{3.3}$$

for **any** constants $x_L < 0 < x_R$. Hence the solutions to (3.1) are "regular" solutions of the conservation law

$$u_t + \left(\frac{1}{2}\,u^2\right)_x = 0, \tag{3.4}$$

away from the position $x = 0$ of the point source, and have a **discontinuity at $x = 0$ satisfying**

$$u_R^2 - u_L^2 = 2, \tag{3.5}$$

where u_L (respectively u_R) is the value of the solution on the left (respectively on the right) side of the discontinuity. In addition, **the discontinuity at $x = 0$ should satisfy causality:**

> *Every point in space time should be connected, via a single characteristic, to a point in the past where data is prescribed.* (3.6)

This restricts the solutions to (3.5). **Not all pairs $(u_L,\, u_R)$ satisfying (3.5) are acceptable for the solutions to (3.1).** Below we show that **the allowed pairs at $x = 0$ are:**

- **P1.** $\boldsymbol{0 < u_L < u_R = \sqrt{u_L^2 + 2}}$. The characteristics enter the discontinuity from the left, and exit on the right with a larger value for u. See figure 3.1.
- **P2.** $\boldsymbol{u_R = -\sqrt{u_L^2 + 2} < 0 < u_L}$. The characteristics on both sides of the discontinuity enter it. As shown below, this corresponds to a shock coalescing with the discontinuity. See figure 3.1.

[4] Expansion fans are regions where all the characteristics emanate from a single point in space time.

P3. $\boldsymbol{0 = u_L < u_R = \sqrt{2}}$. The characteristics on the left of the discontinuity are parallel to it. Characteristics carrying the value $u = \sqrt{2}$ exit the discontinuity on the right. See figure 3.1.

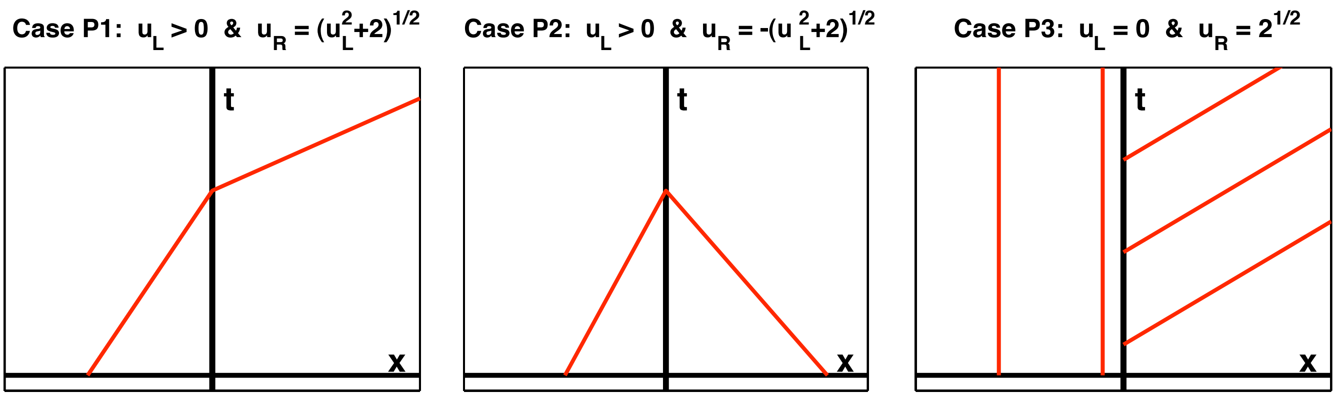

Figure 3.1: Plot of a few typical characteristic curves for the cases in items **P1 – P3.**

P4. $\boldsymbol{u_R = -\sqrt{2} < u_L = 0}$. The characteristics enter the discontinuity from the right, and do not come out. On the left the characteristics are parallel to the discontinuity. See figure 3.2.

As shown below, this case corresponds to either of the following two scenarios:

(i) The characteristics enter the point source region from the right, their speed is brought up to zero by the source, and then they remain trapped on the very left edge of the source.

(ii) There is a shock wave right along the right side of the point source — see remark **3.3**.

P5. $\boldsymbol{u_R = -\sqrt{u_L^2 + 2} < u_L < 0}$. The characteristics enter the discontinuity from the right, and exit on the left with a larger value for u. See figure 3.2.

The excluded pairs at $x = 0$ are:

Ex. $\boldsymbol{u_L < 0 < u_R = \sqrt{u_L^2 + 2}}$. Characteristics emanate from the discontinuity towards both the left and right sides. See figure 3.2.

Clearly, **P1 – P5** and **Ex** cover all the possible solutions to (3.5).

In order to understand why the cases in **P1 – P5** are allowed, while the case in **Ex** is excluded, it is *convenient to think of equation (3.1) as the* $\epsilon \downarrow 0$ *limit of the equation*

$$u_t + \left(\frac{1}{2}\,u^2\right)_x = \frac{1}{\epsilon}\,f\left(\frac{x}{\epsilon}\right), \tag{3.7}$$

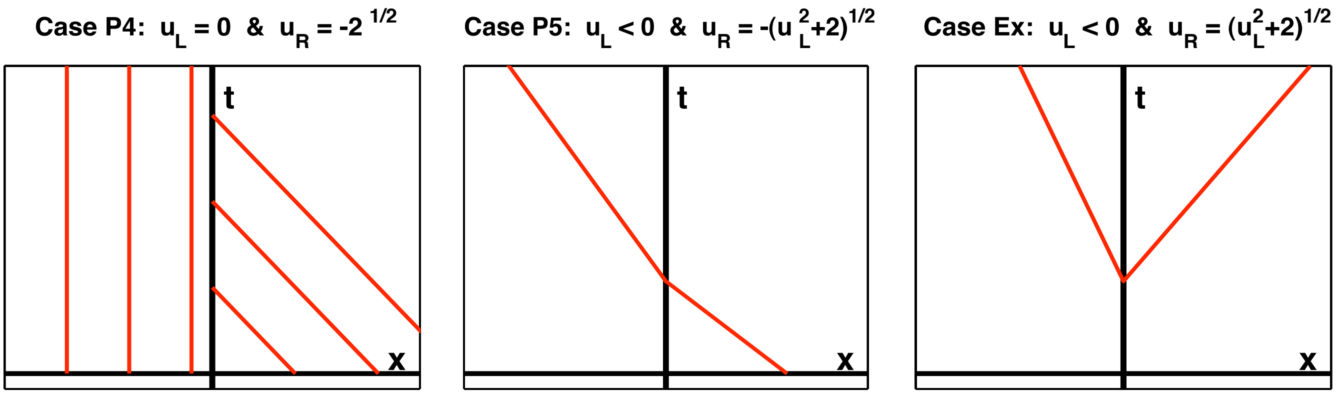

Figure 3.2: Plot of a few typical characteristic curves for the cases in items **P4 – P5** and **Ex.**

where $f = f(z)$ is a smooth function with the properties: (i) $f > 0$ for $-1 < z < 1$. (ii) $f = 0$ for either $z \leq -1$ or $z \geq 1$. (iii) $\int f(z)\,dz = 1$. The advantage of doing this is that, while the characteristic form for (3.1)

$$\frac{du}{dt} = \delta(x) \quad \text{along} \quad \frac{dx}{dt} = u \tag{3.8}$$

does not have a clear meaning at $x = 0$, the characteristic form for (3.7)

$$\begin{aligned} \frac{du}{dt} &= \frac{1}{\epsilon} f\left(\frac{x}{\epsilon}\right) \quad &\text{along} \quad \frac{dx}{dt} &= u, \\ \text{or, equivalently,} \quad \frac{du}{d\tau} &= f(z) \quad &\text{along} \quad \frac{dz}{d\tau} &= u, \end{aligned} \tag{3.9}$$

(where $\boldsymbol{z = x/\epsilon}$ and $\boldsymbol{\tau = t/\epsilon}$) makes perfect sense for any $\epsilon > 0$. Hence, we can study what the characteristics do in this case, and then see what happens in the $\epsilon \downarrow 0$ limit. Below we use this approach to show that the cases in **P1 – P5** are allowed, while the case in **Ex** must be excluded.

-1- Case P1. Consider characteristics that enter the interval $\boldsymbol{I_0 = \{x| -\epsilon < x < \epsilon\}}$ from the left (necessarily with a value $u = u_L > 0$), and traverse I_0. Clearly, u increases as this happens. Hence, after a while (provided that they are not intersected by a shock) the characteristics exit on the right, with a larger value $u = u_R$. This is precisely the scenario of case **P1** — note that, in the limit $\epsilon \downarrow 0$, the characteristics cross I_0 in zero time.

The magnitude of the jump in u as the characteristics cross I_0 follows from noticing that (3.9) yields, for $u \neq 0$,

$$\frac{d}{dx}\left(\frac{1}{2}u^2\right) = \frac{1}{\epsilon} f\left(\frac{x}{\epsilon}\right). \tag{3.10}$$

Upon integrating this from $x = -\epsilon$ to $x = \epsilon$, equation (3.5) follows.

-2- Case P3. Consider now characteristics starting somewhere inside I_0, with $u = 0$. Then both u and x increase with time. After a while, the characteristics exit on the right, with $u = u_1 > 0$.

Let Δt be the time that a characteristic spends inside I_0. What can we say about $\boldsymbol{\Delta t}$? If the characteristic starts close to $x = \epsilon$, then Δt is small. On the other end, if it starts close to $x = -\epsilon$, u and x initially grow very slowly (since $f(z)$ is small near $z = -1$) resulting in a large Δt. Thus $\boldsymbol{\Delta t}$ can take any value in the range $\mathbf{0 < \Delta t < \infty}$. Furthermore, as $\epsilon \downarrow 0$, the characteristic has to start closer and closer to $x = -\epsilon$ to prevent Δt from vanishing — but the full range $0 < \Delta t < \infty$ persists in the limit.

What can we say about $\boldsymbol{u_1}$? It is difficult to say what u_1 is when $\epsilon > 0$. However, in the limit $\epsilon \to 0$, if $\Delta t > 0$, integration of (3.10) from[5] $x = -\epsilon$ to $x = \epsilon$, yields $u_1 = \sqrt{2}$.

Thus, the situation in **case P3 can be interpreted as follows:** The value $u = 0$ to the left of the discontinuity at the origin extends all the way to the left "inside" boundary of the discontinuity. Characteristics along the left "inside" boundary of the discontinuity stay there — with zero speed — for some arbitrary time Δt, at which point they experience an infinite acceleration, and exit to the right of the discontinuity with the value $u = \sqrt{2}$. Note that:

$$\textit{The causality restriction in (3.6) is NOT violated,} \tag{3.11}$$

even though, at first sight, it would seem as if this case does not satisfy causality, with information being injected from the discontinuity at $x = 0$ into the solution.

-3- Case P5. Consider characteristics that enter the interval I_0 from the right (necessarily with a value $u = u_R < 0$), and traverse I_0 right to left. Then u increases as this happens, and the characteristics slow down in its progress towards the left side of I_0. However, if $u = u_R$ is negative enough — $u_R < u_C < 0$, for some critical value u_C — the characteristics make it to the other side (provided that they are not intersected on their way there by a shock), and exit on the left with some (still negative) value $u = u_L$. Using (3.10) one can see that $u_C = -\sqrt{2}$, and that (3.5) applies if the characteristics make it across I_0.

-4- Case P4. Consider the same situation as in the prior case, but assume now that the value with which the characteristics enter I_0 from the right is "critical", $u_R = u_C = -\sqrt{2}$. In this

[5] For $\Delta t > 0$, the characteristic has to start at $x = -\epsilon$ in the limit $\epsilon \to 0$.

case the characteristics slow down as they approach[6] $x = -\epsilon$, with $u \uparrow 0$ as this happens. The characteristics never make it out of I_0, and stay trapped there forever. See remark **3.3**.

At this point, a natural question to ask is: what happens if characteristics enter I_0 from the right, with a value $u_C < u_R < 0$? For the answer, see remark **3.2** below.

-5- Case P2. In all the prior cases we considered characteristics entering I_0 from the left, or the right, and proceeding forward in time without encountering characteristics coming the opposite way (thus triggering the need to insert a shock wave). Consider now what happens when characteristics enters I_0 from the left with some value $u = u_L > 0$, and another set enters from the right carrying $u = u_R < 0$. Then a shock will have to occur inside I_0, where both sets of characteristics terminate. However, if this shock does not have zero speed, it will exit to the left or right (in zero time in the limit $\epsilon \downarrow 0$) — changing the value of u on the left (or right) of the discontinuity at $x = 0$ to a value different from u_L (or u_R). Thus this scenario can happen in the $\epsilon \downarrow 0$ limit only if the shock speed is zero. From conservation, this corresponds to the case where u_L and u_R are related by (3.5) — see remark **3.1** below.

-6- Case Ex. For this case to apply, the characteristics would have to start inside the interval I_0, travel for a while staying inside — even in the limit $\epsilon \downarrow 0$, and then exit to both left and right. However: (i) u increases while a characteristic stays within I_0. (ii) For a characteristic to stay inside I_0 for a finite amount of time, in the limit $\epsilon \downarrow 0$, it has to start with $u = 0$ — if $u < 0$ (respectively $u > 0$) it exits immediately to the left (respectively to the right). Hence, a characteristic that travels "inside" the discontinuity for any finite period of time, can only exit to the right. This shows that this case is not possible. This case violates causality, as it requires characteristics to be created "out of nowhere" on the left boundary of the discontinuity.

Remark 3.1 Rankine-Hugoniot conditions for shocks with a point source tracking them. *Consider the situation where a shock appears in a conservation equation for some quantity ρ, and that (by some means) a point source of ρ is added right at the location of the shock. The relevant equation is then*

$$\rho_t + q_x = A(t)\,\delta(x - x_s(t)), \tag{3.12}$$

[6] Assuming that they do not encounter a shock that "terminates" them.

where $A(t)$ is the amount of conserved "stuff" that is deposited at the shock position per unit time, $x = x_s(t)$ is the shock position, and q is the flux. Write now the integral form of this equation

$$\frac{d}{dt}\int_c^d \rho\, dx = q(c,\, t) - q(d,\, t) + A, \tag{3.13}$$

where $c < x_s < d$ are some arbitrary (fixed) points. Then: **(i)** *Write* $\int_c^d \rho\, dx = \int_c^{x_s} \rho\, dx + \int_{x_s}^d \rho\, dx$. **(ii)** *Note that ρ is "nice" on each side of the shock, and satisfies $\rho_t + q_x = 0$.* **(iii)** *Take the time derivative of the integral in (3.13) using (i), and the standard rules for taking derivatives of integrals with variable boundaries.* **(iv)** *Use (ii) to simplify the result of (iii). This yields:*

$$\frac{dx_s}{dt} = \frac{q_L - q_R}{\rho_L - \rho_R} + \frac{A}{\rho_L - \rho_R}, \tag{3.14}$$

where the subscript L indicates values immediately to the left of the shock, and the subscript R indicates values immediately to the right. Notice that, with $dx_s/dt = 0$, $u = \rho$, $q = (1/2)\,u^2$, and $A = 1$, this reduces to (3.5).

Remark 3.2 *Let us now go back to items* **-3-** *and* **-4-** *above, and ask the question:* What happens if the characteristics enter I_0 from the right, with a value $\boldsymbol{u = u_R}$ such that $\boldsymbol{-\sqrt{2} = u_C < u_R < 0}$? *Then, somewhere inside I_0 the characteristics reverse direction, start moving to the right, and cross the characteristics (from later times) still moving left. Hence a shock is required. In the limit $\epsilon \downarrow 0$, this shock will exit to the right in zero time, and change the value of the solution to the right of $x = 0$ to some other value — hence* this scenario is NOT allowed in the limit $\epsilon \downarrow 0$.

Question: Why does the shock exit to the right? *Because, if it exits to the left, it will not be able to suppress the crossing of characteristics that it is supposed to prevent (which happens inside I_0). Neither can it stay inside I_0, because, just as in item* **-5-**, *this requires that (3.5) be satisfied for some $u_L > 0$. But there is no such u_L if $-\sqrt{2} < u_R < 0$!*

Remark 3.3 *It is interesting to note that* the case in item **P4** can occur with a different scenario from the one used above to show that this was an allowed case. *Namely, imagine the situation in item* **P3**, *with characteristics running along the left edge of the discontinuity, till they experience an infinite acceleration, and exit to the right of the discontinuity with a value $u = \sqrt{2}$. However, say that right there they encounter characteristics arriving at the discontinuity from the right, carrying the value $-\sqrt{2}$. The result is a* shock wave, with speed zero, running along the right edge of the discontinuity.

3.2 Answer: Riemann Problems (problem 02).

The characteristics for (3.1), away from $x = 0$ are given by .... $\boldsymbol{\dfrac{du}{dt} = 0}$ along $\boldsymbol{\dfrac{dx}{dt} = u.}$ Thus they are straight lines with slope $= u =$ constant.

At **rarefaction fans** all the characteristics emanate from a single point (say $(x,\, t) = (0,\, 0)$), hence

$$u = f \quad \text{and} \quad x = t\, f, \quad \Longrightarrow \quad u = \frac{x}{t}, \quad \text{in some region} \quad t\, f_1 < x < t\, f_2, \tag{3.15}$$

where $f_1 \leq f_2$ are constants. At shocks the **Rankine-Hugoniot jump condition,** and the **Entropy condition** reduce to

$$\frac{dx_s}{dt} = \frac{1}{2}\,(u_L + u_R) \quad \text{and} \quad u_L > u_R, \tag{3.16}$$

where u_L is the state immediately to the left of the shock, and u_R is the state immediately to the right of the shock. Putting this, and the stuff in items **P1-P5** and item **Ex**, all together, we arrive at the solution listed below, in items $\boldsymbol{A_1}$-$\boldsymbol{A_6}$, to the Riemann problem in (3.1 – 3.2). See figure 3.3.

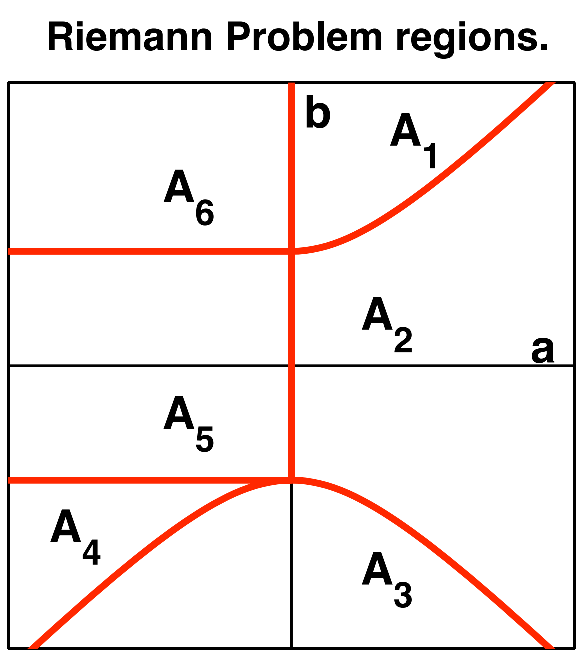

Figure 3.3: Regions of validity for the cases in the Riemann Problem solution. The thick lines, separating the regions, have the equations: (1) $a = 0$ and $b \geq -\sqrt{2}$. (2) $a \geq 0$ and $b = \sqrt{a^2 + 2}$. (3) Any a and $b = -\sqrt{a^2 + 2}$. (4) $a \leq 0$ and $b = -\sqrt{2}$. (5) $a \leq 0$ and $b = \sqrt{2}$.

$\boldsymbol{A_1}$. **Case $\boldsymbol{a \geq 0}$ and $\boldsymbol{b \geq \sqrt{a^2+2}}$.** *Rarefaction to the right of the origin. At the origin, cases:* **P1** *if* $\boldsymbol{a > 0}$, *and* **P3** *if* $\boldsymbol{a = 0}$.

$$u = \begin{cases} a & \text{for} \qquad\qquad\qquad\quad x \;<\; 0, \\ \sqrt{a^2+2} & \text{for} \quad 0 \qquad\quad\;\; < \; x \;\leq\; \sqrt{a^2+2}\,t, \\ \dfrac{x}{t} & \text{for} \quad \sqrt{a^2+2}\,t \;\leq\; x \;\leq\; b\,t, \\ b & \text{for} \quad b\,t \qquad\quad\;\; \leq\; x. \end{cases} \tag{3.17}$$

$\boldsymbol{A_2}$. **Case $\boldsymbol{a \geq 0}$ and $\boldsymbol{-\sqrt{a^2+2} \leq b < \sqrt{a^2+2}}$.** *Shock to the right, or at the origin. At the origin, cases:* **P1** *if* $\boldsymbol{(a > 0,\; b > b_l)}$, **P2** *if* $\boldsymbol{(a > 0,\; b = b_l)}$, **P3** *if* $\boldsymbol{(a = 0,\; b > b_l)}$, *and* **P4** *if* $\boldsymbol{(a = 0,\; b = b_l)}$, *where* $\boldsymbol{b_l = -\sqrt{a^2+2}}$.

$$u = \begin{cases} a & \text{for} \qquad\qquad x \;<\; 0, \\ \sqrt{a^2+2} & \text{for} \quad 0 \;<\; x \;<\; s\,t, \\ b & \text{for} \quad s\,t \;<\; x, \end{cases} \tag{3.18}$$

where $s = \dfrac{1}{2}\left(b + \sqrt{a^2+2}\right)$ is the shock speed.

$\boldsymbol{A_3}$. **Case $\boldsymbol{b < -\sqrt{2}}$ and $\boldsymbol{-\sqrt{b^2-2} < a \leq \sqrt{b^2-2}}$.** *Shock to the left, or at the origin. At the origin, cases:* **P5** *if* $\boldsymbol{a < \sqrt{b^2-2}}$, *and* **P2** *if* $\boldsymbol{a = \sqrt{b^2-2}}$.

$$u = \begin{cases} a & \text{for} \qquad\qquad x \;<\; s\,t, \\ -\sqrt{b^2-2} & \text{for} \quad s\,t \;<\; x \;<\; 0, \\ b & \text{for} \quad 0 \;<\; x, \end{cases} \tag{3.19}$$

where $s = \dfrac{1}{2}\left(a - \sqrt{b^2-2}\right)$ is the shock speed.

$\boldsymbol{A_4}$. **Case $\boldsymbol{b \leq -\sqrt{2}}$ and $\boldsymbol{a \leq -\sqrt{b^2-2}}$.** *Rarefaction fan to the left of the origin. At the origin, cases:* **P5** *if* $\boldsymbol{b < -\sqrt{2}}$, *and* **P4** *if* $\boldsymbol{b = -\sqrt{2}}$.

$$u = \begin{cases} a & \text{for} \qquad\qquad\qquad\quad x \;\leq\; a\,t, \\ \dfrac{x}{t} & \text{for} \quad a\,t \qquad\qquad\;\; \leq\; x \;\leq\; -\sqrt{b^2-2}\,t, \\ -\sqrt{b^2-2} & \text{for} \quad -\sqrt{b^2-2}\,t \;\leq\; x \;<\; 0, \\ b & \text{for} \quad 0 \qquad\qquad\;\;\; <\; x. \end{cases} \tag{3.20}$$

$\boldsymbol{A_5}$. **Case $\boldsymbol{a \le 0}$ and $\boldsymbol{-\sqrt{2} \le b < \sqrt{2}}$.** *Shock to the right, or at the origin. Rarefaction fan to the left of the origin. At the origin, cases:* **P3** *if* $\boldsymbol{b > -\sqrt{2}}$, *and* **P4** *if* $\boldsymbol{b = -\sqrt{2}}$.

$$u = \begin{cases} a & \text{for} \quad x \le a\,t, \\ \dfrac{x}{t} & \text{for} \quad a\,t \le x < 0, \\ \sqrt{2} & \text{for} \quad 0 < x < s\,t, \\ b & \text{for} \quad s\,t < x, \end{cases} \tag{3.21}$$

where $s = \dfrac{1}{2}\,(\sqrt{2} + b)$ is the shock speed.

$\boldsymbol{A_6}$. **Case $\boldsymbol{a \le 0}$ and $\boldsymbol{b \ge \sqrt{2}}$.** *Rarefaction fans both to the right and left, of the origin. At the origin, case* **P3**.

$$u = \begin{cases} a & \text{for} \quad x \le a\,t, \\ \dfrac{x}{t} & \text{for} \quad a\,t \le x < 0, \\ \sqrt{2} & \text{for} \quad 0 < x \le \sqrt{2}\,t, \\ \dfrac{x}{t} & \text{for} \quad \sqrt{2}\,t \le x \le b\,t, \\ b & \text{for} \quad b\,t \le x. \end{cases} \tag{3.22}$$

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4 KdV-Burgers Equation (problem 01).

4.1 Statement: KdV-Burgers Equation (problem 01).

Consider the p.d.e. (in a-dimensional variables)

$$u_t + \left(\frac{1}{2}\,u^2\right)_x = \nu\,u_{xx} + \mu\,u_{xxx}, \quad \text{where } \nu > 0 \text{ and } \mu \neq 0 \text{ are both small.} \tag{4.1}$$

This p.d.e. has been proposed as a simple model for the structure of bores in shallow water.[7]

Your task is: Study the behavior of the physically meaningful traveling wave solutions of this equation, as a function of the equation parameters ν, μ, and any wave parameters (e.g.: wave amplitude, speed, etc.). Are there solutions that approach constants as $x \pm \infty$? If so, how are the

[7] See §13.15 of Whitham's book: *Linear and nonlinear waves.*

constants approached (monotone, oscillations)? Are the constants equal, or different? Are there periodic, oscillatory solutions? Make plots sketching the types of solutions that are possible.

Note: to be **physically meaningful**, a solution has to be **bounded.**

HINT. Exploit problem symmetries to reduce the number of free parameters: Because the equation is Galilean invariant, you need only look at time independent solutions (zero propagation speed):

$$\left(\frac{1}{2}\,u^2\right)_x = \nu\, u_{x\,x} + \mu\, u_{x\,x\,x}, \tag{4.2}$$

This can be integrated to

$$\mu\, u_{x\,x} = -\nu\, u_x + \frac{1}{2}\,u^2 + \kappa, \tag{4.3}$$

where κ is a constant. By a re-scaling of the dependent and independent variables, this last equation can be reduced to one of the following three cases, each having a single parameter

$$u_{x\,x} = -\delta\, u_x + \frac{1}{2}\,u^2 + \sigma, \quad \text{where } \sigma = \pm 1 \text{ or } \sigma = 0, \text{ and } 0 < \delta < \infty. \tag{4.4}$$

Study the physically relevant solutions for these equations, as δ varies.

Remark 4.1 *There is no explicit solution for (4.4). One approach is to write (4.4) as a phase plane system, and find the critical points and their type, the null-clines, etc. Then use this information to deduce the phase portrait for the equation. From this the answer to the problem should follow.*

4.2 Answer: KdV-Burgers Equation (problem 01).

Equation (4.1) is *Galilean invariant*: if $u = u(x,\,t)$ solves (4.1), then $\tilde{u} = u(x-g,\,t)+\dot{g}$ is also a solution — for any function $g = g(t)$. Hence, when studying traveling wave solutions, it is enough to look at the zero velocity case — namely, to equation (4.2), which can be integrated to (4.3).

Introduce

$$\boldsymbol{u = \alpha\,\tilde{u}(\tilde{x}),} \quad \text{where} \quad \boldsymbol{\tilde{x} = \beta\,x,} \quad \text{and} \tag{4.5}$$

-1- For $\boldsymbol{\kappa \neq 0}$: $\boldsymbol{\alpha = \mathrm{sign}(\mu)\,|\kappa|}$ and $\boldsymbol{\beta = \mathrm{sign}(\mu)\,\sqrt{|\kappa|/|\mu|}}$.

-2- For $\boldsymbol{\kappa = 0}$: $\boldsymbol{\alpha = \nu^2/\mu}$ and $\boldsymbol{\beta = \nu/\mu}$.

Then equation (4.3) takes the form[8] in (4.4) where

[8] **We drop the tildes to simplify the notation.**

-3- For $\boldsymbol{\kappa \neq 0}$: $\boldsymbol{\delta = \nu/\sqrt{|\kappa\,\mu|}}$ and $\boldsymbol{\sigma = \mathrm{sign}(\kappa/\mu)}$.

-4- For $\boldsymbol{\kappa = 0}$: $\boldsymbol{\delta = 1}$ and $\boldsymbol{\sigma = 0}$.

From now on we work with the equation in the form (4.4), hence there should not be any confusion produced by dropping the tildes in the variables, to simplify the notation.

Multiply (4.4) by $2\,u_x$. Then
$$\left(u_x^2 - \frac{1}{3}\,u^3 - 2\,\sigma\,u\right)_x = -2\,\delta\,u_x^2. \tag{4.6}$$

Hence
$$\textbf{Equation (4.4) has NO nontrivial periodic traveling waves.} \tag{4.7}$$

Proof: If u is a periodic solution of (4.4), then (4.6) yields $\int_{\text{period}} u_x^2\,dx = 0$. Hence u is constant.

From (4.6), $\boldsymbol{\psi = u_x^2 - \frac{1}{3}\,u^3 - 2\,\sigma\,u}$ is non-increasing. Thus the limits $\boldsymbol{\psi(\pm\infty) = \lim_{x\to\pm\infty}\psi(x)}$ must exist, where the limits need not be finite. In the **case where u is a physical solution**

$$\boldsymbol{\infty > \psi(-\infty) > \psi(\infty) > -\infty,} \quad \textbf{if the solution is nontrivial.} \tag{4.8}$$

In particular **both u and u_x are bounded.**

Proof: Since u is bounded, $\psi(\pm\infty) = -\infty$ is impossible. Since ψ is non-increasing, $\psi(\infty) = \infty$ is also impossible — as it would imply $\psi \equiv \infty$. Hence we need only examine the case $\psi(-\infty) = \infty$. But, since u is bounded, this can only happen if either $u_x \to \infty$ or $u_x \to -\infty$, which would contradict the fact that u is bounded. Finally $\psi(-\infty) = \psi(\infty)$ is possible only if $u_x \equiv 0$, which yields u constant.

Furthermore: **when u is a physical solution**

$$\boldsymbol{u_x \to 0} \text{ **and** } \boldsymbol{u \to u_*} \text{ **as** } \boldsymbol{x \to \pm\infty,} \quad \textbf{where } \boldsymbol{u_*} \textbf{ is a constant.} \tag{4.9}$$

To show this notice that (4.6) yields, for $\boldsymbol{\psi = u_x^2 - \frac{1}{3}\,u^3 - 2\,\sigma}$ and $\boldsymbol{u}$, the system

$$\psi_x = -2\,\delta\,(\psi - \Psi(u)) \quad \text{and} \quad u_x = \pm\sqrt{\psi - \Psi(u)}, \quad \text{where} \quad \Psi(u) = -\frac{1}{3}\,u^3 - 2\,\sigma\,u. \tag{4.10}$$

This applies for $\boldsymbol{\psi \geq \Psi(u)}$, and the sign of the square root can switch ONLY along $\boldsymbol{\psi = \Psi(u)}$.

Remark 4.2 *What happens with the solutions to (4.10) along $\psi = \Psi(u)$ follows from (4.4). From the definition of ψ, $\psi = \Psi(u)$ is equivalent to $u_x = 0$. But there (4.4) yields $u_{xx} = \frac{1}{2}u^2 + \sigma$, hence u_x flips sign —* **except** *if $\frac{1}{2}u^2 + \sigma = 0$ — and the solution continues. Note that:*

r2a. *If a solution satisfies (somewhere) $u_x = 0$ and $u = u_*$ — where $\frac{1}{2}u_*^2 + \sigma = 0$ — then $u = u_*$ everywhere. This follows from uniqueness of the initial value problem for (4.4).*

r2b. *The points where $\frac{1}{2}u^2 + \sigma = 0$ are the extrema of Ψ, i.e.: where $\Psi'(u) = 0$.*

Proof of (4.9): From remark **4.2**, what the **solutions to (4.10)** do in the (u, ψ)-plane follows — see figure 4.1. **As x increases:** ψ decreases, with u either increasing or decreasing.[9] This continues till the solution either reaches the curve $\psi = \Psi(u)$, or it diverges — with $\psi \to -\infty$ (non-physical solution). When/if the solution reaches $\psi = \Psi(u)$, if this happens at a point where $\Psi'(u) \neq 0$, the solution "bounces" of the curve $\psi = \Psi(u)$ — with u_x flipping sign, and the process continues. A non-trivial solution can reach $\psi = \Psi(u)$ at a point where $\Psi'(u) = 0$ only as $x \to \infty$ — otherwise it would be identically constant, as follows from **r2a**. In fact, a non-trivial physical solution has to approach such a point as $x \to \infty$, for the alternative is $\psi \to -\infty$ (non-physical). **An entirely similar argument applies for x decreasing.**

The proof above not only justifies (4.9), it also shows that

$$\textbf{The constant } u_* \textbf{ in (4.9) must satisfy } u_*^2 + 2\,\sigma = 0. \tag{4.11}$$

Hence:

$$\textbf{There is no physical solution if } \sigma = 1. \tag{4.12}$$

Furthermore:

$$\textbf{There is no non-trivial physical solution if } \sigma = 0. \tag{4.13}$$

This second result follows because in this case there is only one u_*, so that (4.8) cannot be satisfied.

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[9] Depending on which sign for the square root is selected.

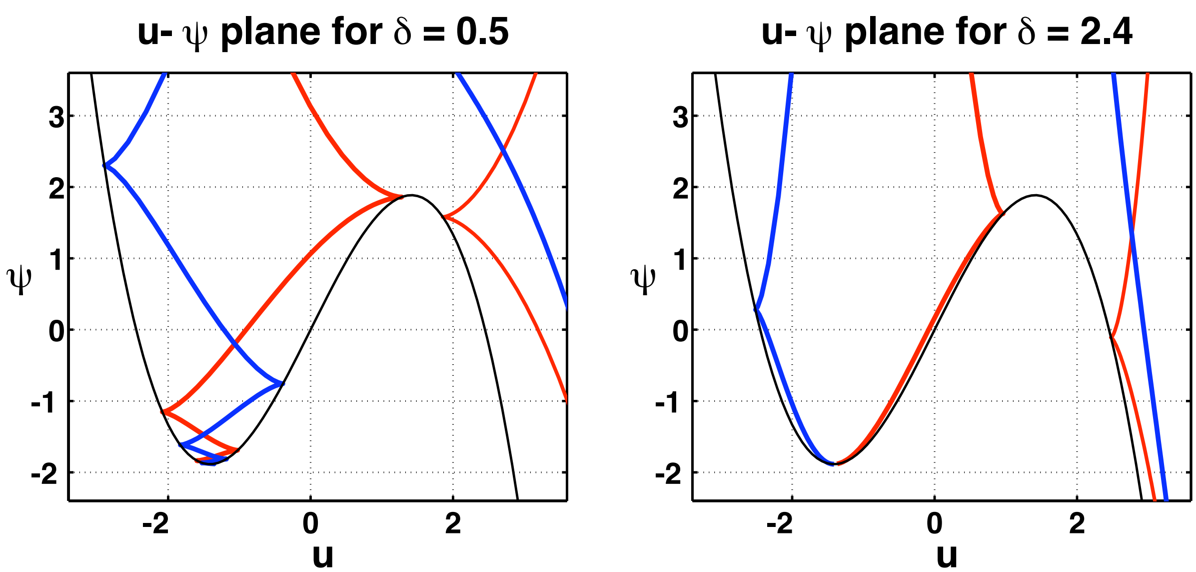

Figure 4.1: Solutions to equation (4.4), for $\sigma = -1$, as they behave in the $(\psi,\, u)$-plane — see (4.10). On the left panel δ is small, and one can see the solutions bouncing back and forth inside the dip in the $\psi = -(1/3)\,u^3 + u$ curve. On the right panel, δ is larger, and the solutions tend to stay close to the curve $\psi = -(1/3)\,u^3 + u$. This is easy to see from the equation for ψ in (4.10): if δ is large, ψ decays rapidly towards $\psi = -(1/3)\,u^3 + u$. The critical value for δ at which the behavior of the solutions switches is $\delta_c = \sqrt{4\,\sqrt{2}} \approx 2.38$.

Case $\sigma = -1$.

In this case there are two possible values for u_*, given by $u_* = \sqrt{2}$ and $u_* = -\sqrt{2}$. From the prior arguments, **if there is a physical solution, it must connect $u = \sqrt{2}$ (with $u_x = 0$) at $x = -\infty$ to $u = -\sqrt{2}$ (with $u_x = 0$) at $x = \infty$. Such a solution exists, and is unique (up to translations).**

Existence: The curve $\psi = \Psi(u)$ has a local minimum at $u = -\sqrt{2}$, with $\Psi = -4\,\sqrt{2}/3$, and a local maximum at $u = \sqrt{2}$, with $\Psi = 4\,\sqrt{2}/3$. Consider a solution to (4.10), starting at $x = -\infty$ from the local maximum, with the negative square root sign — one of the curves on the left panel of figure 4.1 is very close to such a solution. It is easy to see, using arguments like those after remark **4.2**, that this solution is trapped inside the bowl shaped region above the local minimum of Ψ. Hence, it must reach the local minimum as $x \to \infty$. This argument proves existence, except for a little detail that must be clarified: is there such a thing as *"a solution for (4.10) starting at $x = -\infty$, from the local maximum, with ..."?* Below we show that the answer is yes, by identifying these solutions with the unique orbit leaving a saddle point in the direction of decreasing u.

Uniqueness: Equation (4.4), for $\sigma = -1$, can be written as the phase plane system

$$u_x = v \quad \text{and} \quad v_x = -\delta\, v + \frac{1}{2}\, u^2 - 1, \tag{4.14}$$

This system has two critical points: $C_1 = (u,\, v) = (\sqrt{2},\, 0)$ and $C_2 = (u,\, v) = (-\sqrt{2},\, 0)$ — which correspond to the local maximum and minimum of Ψ. It is easy to see that C_1 is a saddle point, so that there is a single orbit leaving C_1 towards $u < \sqrt{2}$. In terms of ψ and u, the solutions corresponding to this orbit are the ones that the existence argument uses. Since the orbit is unique, there is a unique (up to translation) traveling wave solution. **Note:** That this saddle orbit ends up at C_2 as $x \to \infty$ is not easy to show using (4.14). This is why we introduced (4.10).

Finally: What is the nature of the critical point C_2? It is easy to see that

S1. If $\delta > \delta_c = 2^{5/4}$, C_2 is a node. In this case the traveling wave solution connecting $u = \sqrt{2}$ at $x = -\infty$ to $u = -\sqrt{2}$ at $x = \infty$ looks like a standard shock connection: a smooth, monotone decreasing, function connecting the states at $\pm\infty$. See figure 4.2.

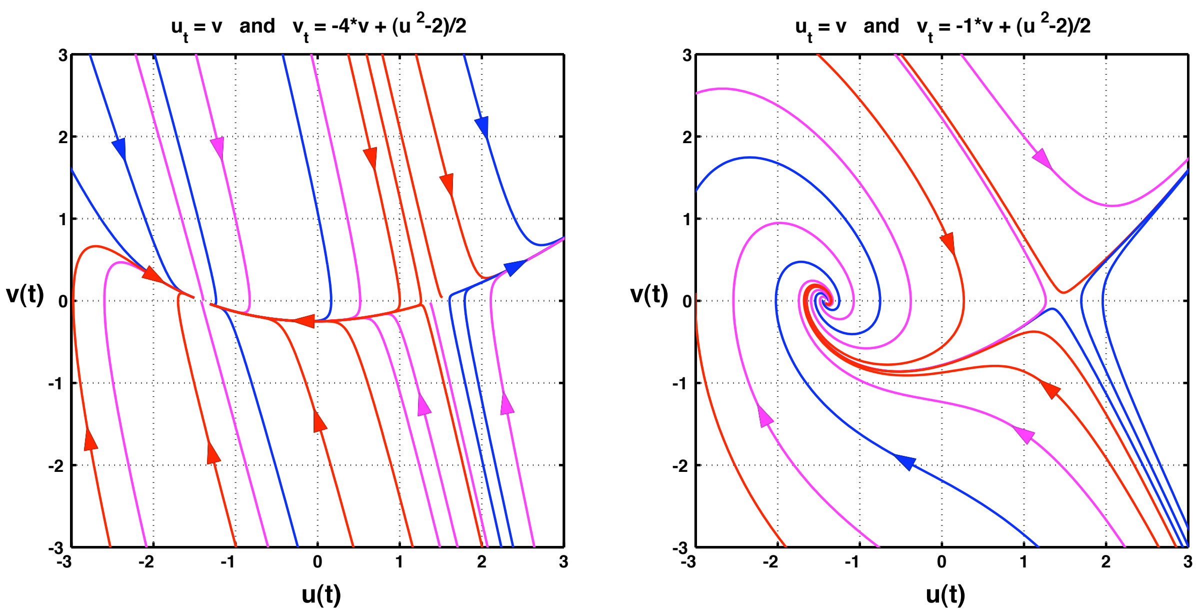

Figure 4.2: Phase plane portrait for (4.14). Left panel: super-critical case $\delta > \delta_c$. Traveling wave is a saddle-node connection. Right panel: sub-critical case $\delta < \delta_c$. Traveling wave oscillates as $x \to \infty$.

S2. If $\delta < \delta_c = 2^{5/4}$, C_2 is a spiral point. In this case the traveling wave solution connecting $u = \sqrt{2}$ at $x = -\infty$ to $u = -\sqrt{2}$ at $x = \infty$ oscillates as it approaches its limit at ∞. Instead of a standard shock connection, this is now an **undulating shock, or bore.**

Note that, to know which way the oscillations occur in the "physical" variables, you must transform these results using (4.5). See figure 4.2.

Remark 4.3 *Many of the arguments in this answer could have been made much shorter using well known facts from the theory of phase plane systems. Specifically: Poincaré-Bendixon theory and Index theory. I decided against using them, since such theories are not a normal part of the pre-requisite courses for 18.306. You can learn it if you take 18.385.*

Summary/Conclusions.

We have shown that, **for every $0 < \delta < \infty$ there exists a unique $U = U(z, \delta)$ defined by:**[10]

u1. $\lim_{z \to -\infty} U(z, \delta) = \sqrt{2}$, and $\lim_{z \to \infty} U(z, \delta) = -\sqrt{2}$.

u2. $U(z, \delta) > 0$ for $z < 0$ and $U(0, \delta) = 0$.

u3. $U'' = -\delta U' + \frac{1}{2} U^2 - 1.$ where the **primes** indicate derivatives with respect to z.

See figure 4.3 for typical plots for U.

Let $\delta_c = \sqrt{4\sqrt{2}}$. Then

u4. As $z \to -\infty$, U **approaches $\sqrt{2}$ exponentially** .... $U \sim \sqrt{2} - e^{\lambda_1 (z - z_1)}$, where z_1 is a constant, and $\lambda_1 = \frac{1}{2}\left(\sqrt{\delta^2 + \delta_c^2} - \delta\right) > 0$.

u5. If $\delta > \delta_c$, as $z \to \infty$, U **approaches $-\sqrt{2}$ exponentially** ...... $U \sim -\sqrt{2} + e^{-\lambda_2 (z - z_2)}$, where z_2 is a constant, and $\lambda_2 = \frac{1}{2}\left(\delta - \sqrt{\delta^2 - \delta_c^2}\right) > 0$.

u6. If $\delta < \delta_c$, as $z \to \infty$, U **approaches the limit $-\sqrt{2}$ with exponentially damped oscillations** .... $U \sim -\sqrt{2} + e^{-\delta (z - z_3)/2} \cos\left(\frac{1}{2}\sqrt{\delta_c^2 - \delta^2}\,(z - z_4)\right)$, where z_3 and z_4 are some constants.

[10] Condition **u2** eliminates the translational degree of freedom in traveling waves, resulting in a unique answer.

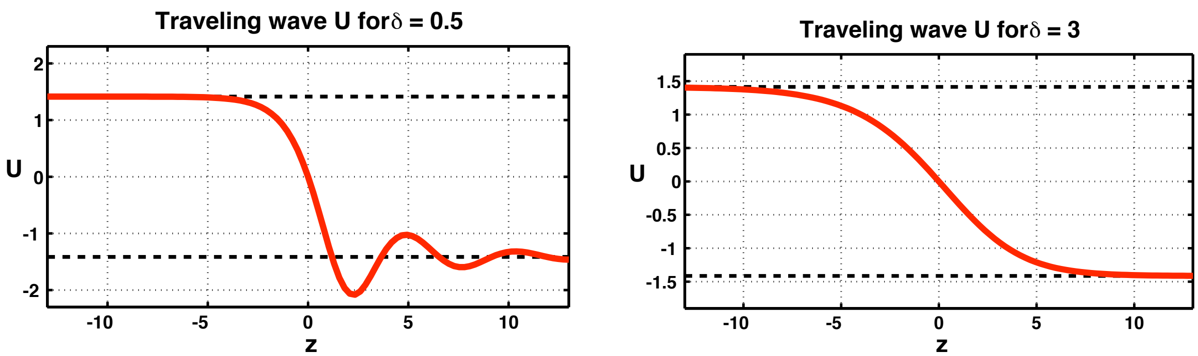

Figure 4.3: Example profiles for $U = U(z,\, \delta)$, as a function of z, for two values of δ. On the left panel, where $\delta = 0.5 < \delta_c$, the solution oscillates as $z \to \infty$. On the right panel, where $\delta = 3 > \delta_c$, the solution is monotone.

Then **all the physically meaningful traveling waves for (4.1) are given by**

$$\boldsymbol{u = \alpha\, U\, (\beta\, (x - s\, t,\, \delta)) + s} \tag{4.15}$$

where, in terms of the limits of the traveling wave as $\boldsymbol{x \to \pm\infty}$, namely $\boldsymbol{u_- > u_+}$, we have

$$\boldsymbol{s = \frac{u_- + u_+}{2}, \quad \alpha = \mathrm{sign}(\mu)\, \frac{[u]}{2\,\sqrt{2}},} \quad \text{and} \quad \boldsymbol{\beta = \mathrm{sign}(\mu)\, \sqrt{\frac{[u]}{2\,\sqrt{2}\,|\mu|}},} \tag{4.16}$$

with $\boldsymbol{[u] = u_- - u_+}$ and $\boldsymbol{\delta = \dfrac{\nu}{\sqrt{2\,|\mu|\,[u]}}\,\delta_c}$.

THE END.