18.306 Advanced Partial Differential Equations with Applications
Fall 2009

MIT 18.306 Fall 2009
Problem set #1
Due Friday October 2

Problem #1

Sove the following problem, using the method of characteristics (compute the characteristics as done in the lectures, then solve for the solution along the characteristics, and then eliminate the characteristic variables to obtain the solution as a function of x and y)..

$(x-y)*u_x + (x+y)*u_y = x^2 + y^2;$

with $u(x, 0) = (1/2)*x^2$ for $1 <= x < exp(2*pi)$.

Answer this question:
Where does this define the solution u? That is: what is the region of the plane characterized by the property that through each point in it there is exactly one characteristic connecting it with the curve where the data is given?

Problem #2

Let rho = rho(x, t) be the density of some conserved quantity, and let q = q(x, t) be the corresponding flux. Then, in the absence of sources

$rho_t + q_x = 0.$ (2.1)

Assume now that, while examining the physical problem behind this equation, you convince yourself that a ``good approximation'' for the flux is

$q = rho + c*rho_x,$ (2.2)

where c is some constant. Substituting (2.2) into (2.1) yields then a pde for rho. What restriction should you impose on the constant c so that the equation is not ill-posed (specifically: it does not have arbitrarily large growth factors for the high frquencies)?

Problem #3
Consider the problem

u_t + u_x = u^2, for t > 0 and -infinity < x < infinity,

with initial condition u(x, 0) = 1/(1 - x + x^2).

Where is the solution defined?
--- Compute, explicitly, the boundary of the region where the solution
    is defined --- t as a function of x, or x as a function of t.
--- Do a plot of the region where the solution is defined.

Problem #4
Consider the problem

u_t + u*u_x = 0, for t > 0 and -infinity < x < infinity,

with initial condition u(x, 0) = F(x) = - arctan(x).

Where is the solution defined?
--- Compute the characteristics, and find the region in t > 0 characterized
    by the property that exactly one characteristic goes trough each point
    in it. Notice that if x = X(zeta, t) is the formula for the
    characteristics [zeta being the label --- X(zeta, 0) = zeta] then, for
    any fixed time, the multiple valued region is in-between the points
    where \partial X / \partial zeta vanihes (justify this, see a-c below).
--- Do a plot of the region in space-time where the characteristics give
    a multiple valued answer.

Note that F = F(x) above has the following properties
(a) G = dF/dx < 0.
(b) G(x) vanishes as |x| ---> infinity.
(c) G has a single minimum.

Problem #5 [this one is tricky]
Consider the problem

u_t + u*u_x = 0, for t > 0 and -infinity < x < infinity,

with initial condition u(x, 0) = F(x).

Assume now that the boundary of the region of multiple values of the solution by characteristics is given by $t = 1+x^2$ (multiple values for $t > 1 + x^2$).

Question: what can you say about F? Can you determine it from the information given?

Important: this looks as if I am asking you to determine the past from the future. Hence, if you could determine F, as requested, causality would be violated.
However, this is not quite so. As we will see later, in cases where multiple values appear, the physically relevant solution does not allow you full access to the boundary of the region of multiple values. In fact, usually you can get only a few points. Hence, by giving you the full curve, I am giving a lot of extra information that is not within the physically relevant solution.

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