18.306 Advanced Partial Differential Equations with Applications
Fall 2009

Answers to Problem Set Number 02 for 18.306 — MIT (Fall 2009)

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1.1 Statement: Linear 1st order PDE (problem 09).

Surface Evolution. The evolution of a material surface can (sometimes) be modeled by a pde. In evaporation dynamics, where the material evaporates into the surrounding environment, consider a surface described in terms of its "height" $h = h(x,\, y,\, t)$ relative to the $(x,\, y)$-plane of reference. Under appropriate conditions, a rather complicated pde can be written[1] for h. Here we consider a (drastically) simplified version of the problem, where the governing equation is

$$h_t = \frac{A}{r}\, h_r, \quad \text{for } r = \sqrt{x^2 + y^2} > 0 \text{ and } t > 0, \quad \text{where } A > 0 \text{ is a constant.} \tag{1.1}$$

Axial symmetry is assumed, so that $h = h(r,\, t)$. Obviously, ***h* should be an even function of** r. This is both evident from the symmetry, and necessary in the equation to avoid singular behavior at the origin. Assume now

$$h(r,\, 0) = H(r^2), \tag{1.2}$$

where H is a smooth function describing a localized bump. Specifically: **(i)** $H(0) > 0$, **(ii)** H is monotone decreasing. **(iii)** $H \to 0$ as $r \to \infty$. **Note that** $h(r,\, 0)$ *is an even function of* r.

1. Using the theory of characteristics, write an explicit formula for the solution of (1.1 – 1.2).
2. Do a sketch of the characteristics in space time — i.e.: $r > 0$ and $t > 0$.
3. What happens with the characteristic starting at $r = \zeta > 0$ and $t = 0$ when $t = \zeta^2/2\,A$?

[1] From mass conservation, with the details of the physics going into modeling the flux and sink/source terms.

4. Show that the resulting solution is an even function of r for all times.

5. Show that, as $t \to \infty$, the bump shrinks and vanishes.

1.2 Answer: Linear 1st order PDE (problem 09).

The characteristic form of equation (1.1) is

$$\frac{dh}{dt} = 0 \quad \text{along the curves} \frac{dr}{dt} = -\frac{A}{r}. \tag{1.3}$$

This yields

$$r = \sqrt{\zeta^2 - 2\,A\,t} \quad \text{and} \quad h = H(\zeta^2), \tag{1.4}$$

for the characteristic that starts (time $t = 0$) at $0 < r = \zeta < \infty$. The characteristics are parabolas pointing downward in space-time, with their "tips" along the time axis. When a characteristic reaches the origin, it exits the domain where the equation is valid, and it ends. See figure 1.1.

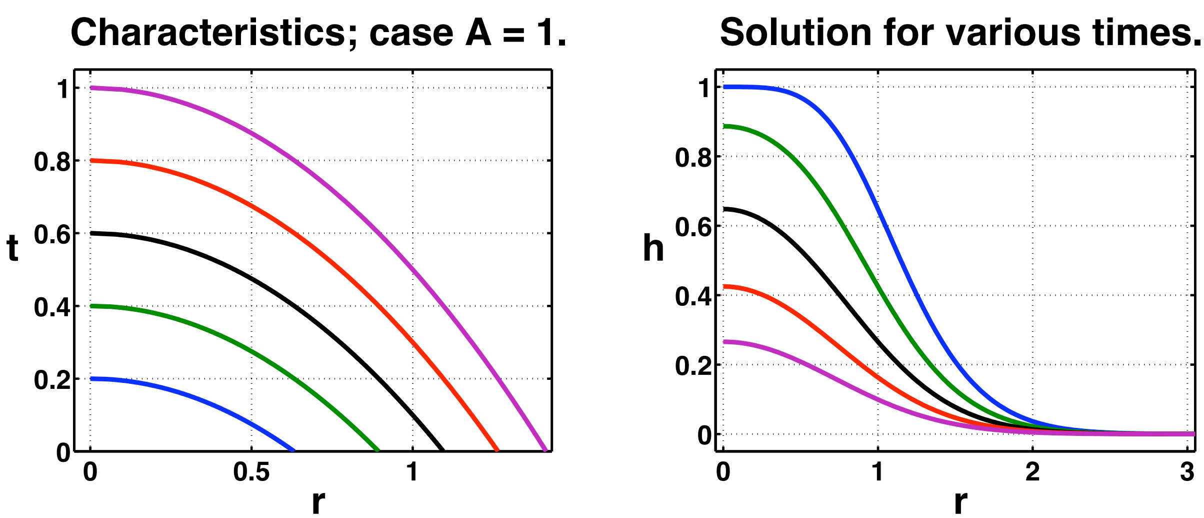

Figure 1.1: Left panel: plot of a few typical characteristic curves for equation (1.1). Right panel: plots of the solution for $H(z) = \text{sech}(z)$, $A = 1$, and times (top to bottom) $t = 0,\ 1/4,\ 1/2,\ 3/4,\ 1$.

From the equation on the left in (1.4), we see that $\zeta^2 = r^2 + 2\,A\,t$. It follows that the solution to the problem in (1.1 – 1.2) is

$$h = H(r^2 + 2\,A\,t). \tag{1.5}$$

Clearly, *this is an even function of* r *for all times.* Furthermore, since H vanishes as it's argument goes to infinity, *the bump described by (1.5) shrinks and vanishes as* $t \to \infty$*.* See figure 1.1.

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2 Quasi-Linear 1st order PDE (problem 02).

2.1 Statement: Quasi-Linear 1st order P.D.E. (problem 02).

Consider the problem

$$u_t + u\, u_x = 0, \quad \text{for } \; t > 0 \quad \text{and} \quad -\infty < x < \infty, \tag{2.1}$$

with initial condition $\boldsymbol{u(x,\, 0) = F(x)}$ for $-\infty < x < \infty$. Assume that the boundary of the region of multiple values for the solution by characteristics is known, and given by the equation

$$t = 1 + x^2, \quad \text{for } -\infty < x < \infty, \tag{2.2}$$

with multiple values in the region $t > 1 + x^2$. **QUESTION:** *What can you say about* F*? Can you determine it from the information given?* **NOTE: this problem is a little tricky!**

Remark 2.1 *This sounds as determining the past from the future: if* F *could be found, causality would be violated. However, this is not quite so. In cases where multiple values appear, the physically relevant solution does not allow access to the multiple values region's boundary. Hence, by giving the full curve, this problem provides extra information not present in the physically relevant solution.*

2.2 Answer: Quasi-Linear 1st order P.D.E. (problem 02).

The solution by characteristics of the problem in (2.1) is given by

$$u = F(\zeta) \quad \text{along the lines} \quad x = \zeta + t\, F(\zeta) = X(\zeta,\, t), \quad \text{where } -\infty < \zeta < \infty. \tag{2.3}$$

Clearly, for each time $t > 0$, the region of multiple values corresponds to those values of x for which the equation for ζ given by $x = X(\zeta, t)$ has more than one solution $\zeta = Z(x, t)$.

Proof: Let $(x,\, t)$ be such that $\zeta_1 \neq \zeta_2$ exist satisfying $x = X(\zeta, t)$. Then $\zeta_1 - \zeta_2 = t\,(u_2 - u_1)$, where $u_j = F(\zeta_j)$. Since $\zeta_1 - \zeta_2 \neq 0$, $u_1 - u_2 \neq 0$.

Hence:

1 If F is continuous, multiple values occur (at any particular time) if and only if X fails to be monotone as a function of ζ. Namely: X must either have local maximums or minimums. Then, in the equation $x = X(\zeta,\, t)$, as x goes down (resp. up) through a value corresponding to a local maximum (resp. minimum), two new solutions (at least) appear.

2 If F is differentiable, a necessary condition for multiple values to occur (at any particular t) is that $X_\zeta = 0$ somewhere. If $X_{\zeta\zeta} \neq 0$ at any of those points, then multiple values occur.

From **1** and **2** we arrive at the following result:

Lemma 2.1 *Let Γ be a curve which is a part of the boundary of the region of multiple values (arising from a differentiable F), which can be described by an equation of the form $x = f(t)$. Then the points in Γ must correspond to values of ζ where $X_\zeta = 0$. In other words,* **Γ must be a part of the curve (parameterized by ζ) described by the equations***, where $G = dF/d\zeta$,*

$$t = -1/G(\zeta) = \tau(\zeta) \quad \textit{and} \quad x = \zeta - F(\zeta)/G(\zeta) = \chi(\zeta), \quad -\infty < \zeta < \infty, \tag{2.4}$$

Proof: $0 = X_\zeta = 1 + t\,G$ leads to the first equation. The second arises from substituting the value of t thus obtained into the equation for X.

Note: *the restriction that, along Γ, x should be a function of t follows because* **1-2** *apply at any fixed t, as x is varied and the number of values for the solution changes. If x is kept fixed, and t is varied, situations can arise where a change in the number of values cannot be associated with the vanishing of X_ζ — we will show an example later in this answer.*

From lemma **2.1** it follows that, if the boundary of the region of multiple values is known, then (at least in principle) it should be possible to find F, since then (2.4) can be used to obtain an o.d.e. for F. In particular, (2.2) leads to[2]

$$0 = 1 + \frac{1}{G} + \left(\zeta - \frac{F}{G}\right)^2, \tag{2.5}$$

which is a nonlinear, first order, o.d.e. for F, since $G = F'$. Hence, the problem has been reduced to that of solving an o.d.e. However, this is easier said than done:

[2] Note that (2.2) yields two curves satisfying the hypothesis of lemma **2.1**. Namely: $x = \pm\sqrt{t-1}$, with $t \geq 1$.

1. Equation (2.5) may be a first order scalar o.d.e., but it is also non-constant coefficients, and with a nasty nonlinearity: Write the equation in the standard form $G = f(F,\, \zeta)$. Then f is multiple valued (it involves a square root), and yields imaginary values unless F and ζ are appropriately restricted. Hence, solving this equation is not trivial.
2. The general solution to (2.5) involves a free constant. Does this mean that there is a one parameter family of possible choices for F, all of them leading to the same region of multiple-values? Or is there some reason that allows only a single (or a few) choices for the parameter?

Rather than attempt to solve (2.5) directly, we will use here an alternative path, which leads to easily solvable equations, and no free parameters.

Using the definition for τ and χ in (2.4), $G = dF/d\zeta$, and $\tau = 1 + \chi^2$, we can write

$$\frac{d\chi}{d\zeta} = F\,\frac{d\tau}{d\zeta} = 2F\chi\,\frac{d\chi}{d\zeta}. \tag{2.6}$$

Hence, either $d\chi/d\zeta = 0$ or $2F\chi = 1$. We analyze these two cases below.

The case $d\chi/d\zeta = 0$.

Clearly, if $\chi(\zeta) \equiv x_0$ is constant, then the *boundary of the region of multiple values is just a point, not a parabola!* Hence, this case is not relevant to the problem at hand. Nevertheless, let us continue the analysis, since having the boundary of the region of multiple values as a single point is a rather puzzling statement; **What does this mean?**

Substituting $\chi \equiv x_0$ into the definition for χ in (2.4) yields

$$\frac{G}{F} = \frac{1}{\zeta - x_0} \quad \Longrightarrow F = g\,(\zeta - x_0) \quad \text{and} \quad G = g = -\frac{1}{\tau}, \tag{2.7}$$

where g is some constant. But $\tau = 1 + \chi^2 = 1 + x_0^2$, so we end up with

$$F(\zeta) = -\frac{\zeta - x_0}{1 + x_0^2}. \tag{2.8}$$

For this initial data, it is easy to eliminate ζ from (2.3), and obtain the following explicit expression for the solution to (2.1)

$$u = \frac{x - x_0}{t - 1 - x_0^2}. \tag{2.9}$$

This is a straight line with slope $1/(t - 1 - x_0^2)$. At $t = 1 + x_0^2$, the line becomes vertical at $x = x_0$, and the whole solution ceases to exist. *There is no "region of multiple values" to speak of* — though, right at $x = x_0$ and $t = 1 + x_0^2$, u takes all possible values.

The case $2F\chi = 1$.

Substituting $\chi = 1/(2\,F)$ and $\tau = 1 + \chi^2 = 1 + 1/(4\,F^2)$ into the definitions for τ and χ in (2.4) yields

$$0 = \left(1 + \frac{1}{4\,F^2}\right) G + 1 \quad \text{and} \quad 0 = \left(\frac{1}{2\,F} - \zeta\right) G + F. \tag{2.10}$$

Using the equation on the left to eliminate G on the equation on the right yields the equivalent set

$$0 = \frac{d}{d\zeta}\left(F - \frac{1}{4\,F} + \zeta\right) \quad \text{and} \quad 0 = -\left(\frac{1}{2\,F} - \zeta\right) + F + \frac{1}{4\,F} = F - \frac{1}{4\,F} + \zeta. \tag{2.11}$$

This has the following two solutions

$$F = \frac{-\zeta \pm \sqrt{1+\zeta^2}}{2}. \tag{2.12}$$

It would seem that we are done now, with F as given by either of the two possibilities in (2.12) being the answer to this problem. But **this is not quite so:** all we know so far is that, if the region of multiple values is given by $t > 1 + x^2$, then F must be as in (2.12). But: is the region of multiple values that corresponds to (2.12) actually $t > 1 + x^2$? This we must still check. Thus the next task is to *calculate directly what the region of multiple values that (2.12) leads to is.*

Consider the case when (the analysis for other case is quite similar)

$$F = \frac{-\zeta + \sqrt{1+\zeta^2}}{2}. \tag{2.13}$$

Then $\boldsymbol{F \sim \frac{1}{4\,\zeta}}$ as $\zeta \to \infty$, $\boldsymbol{F \sim -\zeta - \frac{1}{4\,\zeta}}$ as $\zeta \to -\infty$,

$$\frac{dF}{d\zeta} = -\frac{1}{2} + \frac{\zeta}{2\sqrt{1+\zeta^2}} < 0, \quad \text{and} \quad \frac{d^2F}{d\zeta^2} = \frac{1}{2\,(1+\zeta^2)^{3/2}} > 0. \tag{2.14}$$

Thus F is monotone decreasing, and $G = F'$ is monotone increasing — from $G(-\infty) = -1$ to $G(\infty) = 0$. It is then easy to see that:

A. For $0 \le t < 1$, $x = X(\zeta, t)$ is monotone increasing in ζ, from $X(-\infty,\, t) = -\infty$ to $X(\infty,\, t) = \infty$. Hence, for this time range, the solution u is single valued, and defined for all x. Since F is monotone decreasing, going from ∞ on the left to 0 on the right, so is u. See figure 2.1.

Proof: $G > -1$, thus $X_\zeta = 1 + G(\zeta)\,t > 0$. $X \sim \zeta$ as $\zeta \to \infty$. $X \sim (1-t)\,\zeta$ as $\zeta \to -\infty$.

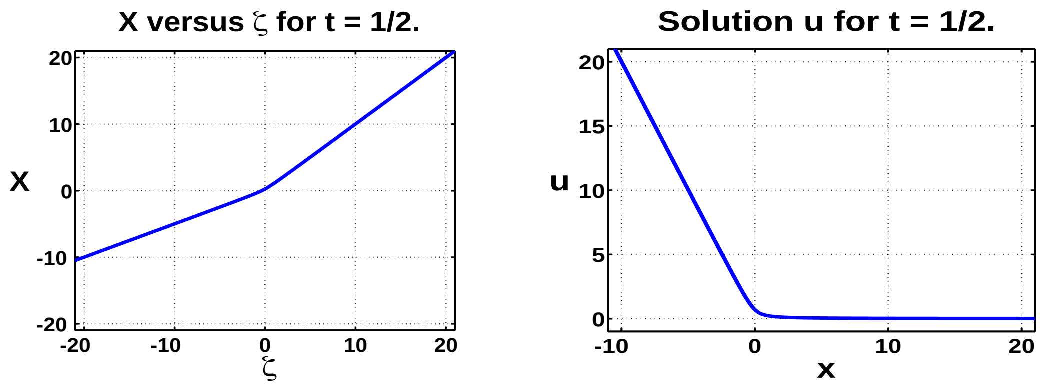

Figure 2.1: Solution by characteristics of (2.1), F as in (2.13). Left: characteristic coordinate $x = X(\zeta,\, t)$ — as in (2.3) — for some fixed time $0 \leq t < t_c = 1$. Right: solution u, for the same time t.

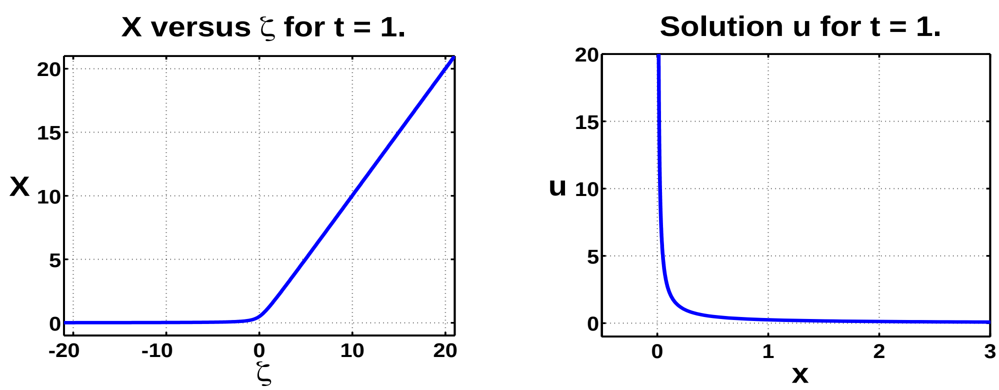

Figure 2.2: Solution by characteristics of (2.1), F as in (2.13). Left: characteristic coordinate $x = X(\zeta,\, t)$ — as in (2.3) — for time $t = t_c = 1$. Right: solution u, for the same time t.

B. For $t = t_c = 1$, $x = X(\zeta, t)$ is monotone increasing in ζ, from $X(-\infty,\, 1) = 0$ to $X(\infty,\, 1) = \infty$. Hence, for this time, the solution u is single valued, but defined for $x > 0$ only. Again, u is monotone decreasing, going from ∞ at $x = 0$ to 0 at $x = \infty$. See figure 2.2.

Proof: : similar to the proof of **A**.

C. For $t > t_c = 1$, $x = X(\zeta, t)$ has a single global minimum somewhere (where $x = x_m$, say), and goes to infinity on either side as $\zeta \to \pm\infty$. Hence, for this time range, the solution u is defined (but double valued) for $x > x_b$ only. Furthermore, backtracking the calculations that lead us to the form for F in (2.13), it is easy to see that $x_m = \sqrt{t-1}$. See figure 2.3.

Proof: similar to the proof of **A**.

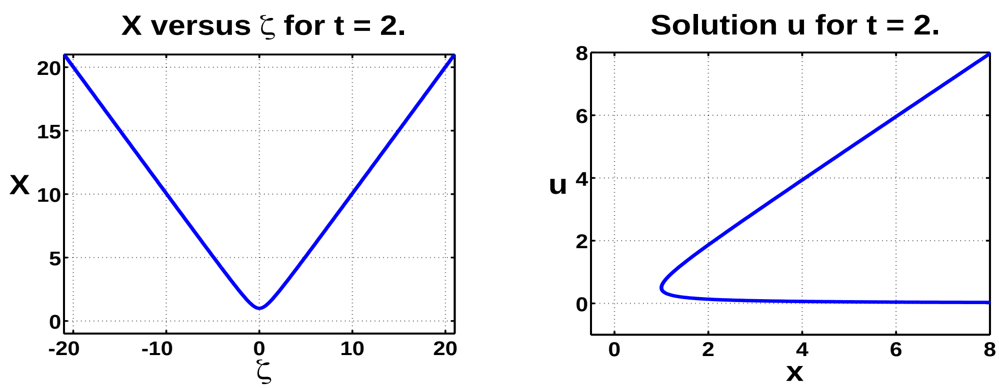

Figure 2.3: Solution by characteristics of (2.1), F as in (2.13). Left: characteristic coordinate $x = X(\zeta,\, t)$ — as in (2.3) — for some fixed time $t > t_c = 1$. Right: solution u, for the same time t.

We conclude that **the region of multiple values corresponding to (2.13) is NOT $t > 1 + x^2$, but the region described by:**[3]

$$t > 1 \quad \text{and} \quad x > \sqrt{t-1}. \tag{2.15}$$

Hence: we only **get 1/2 of the parabola, with the region of multiple values on the wrong side!**.

Remark 2.2 Regarding the solutions to the nonlinear o.d.e. in (2.15). *As a consequence of our "alternative" analysis, we now can write all the solutions to the nonlinear o.d.e. (2.15). These are given by the formulas in (2.8) and (2.12). Note that the solution in (2.8) has a parameter in it, while the solutions in (2.12) do not. This is because the two solutions in (2.12) are a special kind of solution, so called* **envelope solutions** *— see below.*

Finally: consider the initial value problem for (2.5), where $F(0) = F_0$ is given. Then, for any F_0 such that $F_0^2 \neq 1/4$, the equation has two solutions[4] *— which have the form in (2.8). On the other hand, for $F_0 = \pm 1/2$, the solutions are unique, and given by (2.16).*

[3]Note the straight part of the boundary: $t = 1$ and $x \geq 0$, which is not associated with $X_\zeta = 0$. In fact, it arises because of the behavior of F as $\zeta \to -\infty$.

[4]The reason for the two solutions is that (2.5) is, in fact, two equations — since it is quadratic in the derivative, with two possible values of G for almost all values of F (at $\zeta = 0$, the exceptions are $F = \pm 1/2$).

Envelope solutions: Consider a first order o.d.e. for $y = y(x)$

$$f(y,\, dy/dx, x) = 0, \tag{2.16}$$

where f is some smooth function. Assume that $\boldsymbol{y = Y(x,\, s)}$ is a one parameter family of solutions to this equation. Let $\boldsymbol{y = y_e(x)}$ be part of the envelope for the family $y = Y(x,\, s)$. Then:

$$\textbf{The function } \boldsymbol{y = y_e(x)} \textbf{ solves (2.16),} \text{ and it is called an } \textbf{envelope solution.} \tag{2.17}$$

Tasks left to the reader:

T1. Give a proof of (2.17).

T2. By direct substitution, show that (2.8) solves (2.5) for any x_0.

T3. Show that the envelope of the family of curves in (2.8), is given by the two curves in (2.12).

ANSWERS to the tasks left to the reader:

aT1. Proof of (2.17). The envelope of a family of curves has the property that, for each point P in the envelope: (i) There is some curve in the family, say the curve $\mathcal{C}$, such that P also belongs to $\mathcal{C}$. (ii) The envelope and the curve $\mathcal{C}$ are tangent at P.

Applying this to (2.17), we see that there must be some function $s = S(x)$, such that: (i) $y_e(x) = Y(x, S(x))$. (ii) $\dfrac{dy_e}{dx}(x) = \dfrac{\partial Y}{\partial x}(x,\, S(x))$. Hence, since Y satisfies (2.16) for every choice of s, y_e must also solve (2.16).

aT2. If F is given by (2.8), then $\dfrac{F}{G} = \zeta - x_0$. Hence (2.5) reduces to $0 = 1 + \dfrac{1}{G} + x_0^2$, which is obviously true for F as in (2.8).

aT3. The envelope for the family of curves in (2.8) is defined by the two equations

$$F(\zeta) = -\frac{\zeta - x_0}{1 + x_0^2} \quad \text{and} \quad 0 = \frac{1}{1 + x_0^2} + \frac{2\,(\zeta - x_0)\, x_0}{(1 + x_0^2)^2} = \frac{1 + 2x_0\zeta - x_0^2}{(1 + x_0^2)^2}. \tag{2.18}$$

Using the second equation to write x_0 as a function of ζ, and substituting this expression into the first equation, yields (2.12).

3 Linear 1st order PDE (problem 10).

3.1 Statement: Linear 1st order PDE (problem 10).

Integrating factors. **Show that** the pde

$$(a(x,\, y)\, \mu)_y = (b(x,\, y)\, \mu)_x \tag{3.1}$$

is a necessary and sufficient condition guaranteeing that $\mu = \mu(x,\, y) \neq 0$ is an integrating factor for the ode

$$a(x,\, y)\, dx + b(x,\, y)\, dy = 0, \tag{3.2}$$

in any open subset of the plane without holes.

Part II. Assume that $\boldsymbol{a = 3\,x\,y + 2\,y^2}$ and $\boldsymbol{b = 3\,x\,y + 2\,x^2}$**.**
Find an integrating factor for (3.2) — i.e.: obtain a nontrivial solution of (3.1). **Use it to integrate** (3.2), and **write (3.2) in the form**

$$\Phi(x,\, y) = \text{constant}, \quad \text{for some function } \Phi. \tag{3.3}$$

Hint 3.1 *Solving by characteristics (3.1) leads to (3.2), or equivalent, as part of the process — check this! To* get out of this circular situation, note that: *for a and b as above, $\mu = F(x,\, y)$ solves (3.1) iff $\mu = F(y,\, x)$ does. This suggests that you should look for solutions*[5] *invariant under this symmetry; namely: $\mu(x,\, y) = \mu(y,\, x)$. Hence* write $\boldsymbol{\mu = \mu(u,\, v)}$, with $\boldsymbol{u = x + y}$ and $\boldsymbol{v = x\,y}$, *since solutions that satisfy $\mu(x,\, y) = \mu(y,\, x)$ must have this form — see remark* ***3.1****.*

Remark 3.1 *The transformation $(x,\, y) \to (u,\, v)$ is not one to one: it maps the whole xy-plane into the region $v \leq \frac{1}{4}\,u^2$ of the uv-plane, with double valued inverse $x = \frac{1}{2}\left(u \pm \sqrt{u^2 - 4\,v}\right)$ and $y = \frac{1}{2}\left(u \mp \sqrt{u^2 - 4\,v}\right)$. Furthermore:* **(a)** *The two inverses are related by the $x \leftrightarrow y$ switch.* **(b)** *The singular line $u^2 = 4\,v$ corresponds to the line $x = y$.* **(c)** *The map is a bijection between the regions $x \leq y$ and $4\,v \leq u^2$.* **(d)** *The map is a bijection between the regions $x \geq y$ and $4\,v \leq u^2$. From* **(c - d)** *we see that: for* any $\mu = \mu(x,\, y)$*, $\mu = f(u,\, v)$ for $x \leq y$ and $\mu = g(u,\, v)$ for $x \geq y$, for some f and g. Then,* if $\boldsymbol{\mu(x,\, y) = \mu(y,\, x)}$*, $f = g$, so that* $\boldsymbol{\mu}$ has the form $\boldsymbol{\mu = \mu(u,\, v)}$.

Part III. **Why is it that this approach CANNOT be generalized to three variables?** That is, to find integrating factors for equations of the form

$$a(x,\, y,\, z)\, dx + b(x,\, y,\, z)\, dy + c(x,\, y,\, z)\, dz = 0. \tag{3.4}$$

Note: this problem is based on Levine's problem 11 in chapter 9.

[5] Note that you only need **one** nontrivial solution.

3.2 Answer: Linear 1st order PDE (problem 10).

If (3.1) is satisfied in an open subset without holes, then there exists a function $\Phi = \Phi(x,\, y)$ such that

$$\Phi_x = a\,\mu \quad \text{and} \quad \Phi_y = b\,\mu. \tag{3.5}$$

In fact, pick a fixed point in the set, say Q. Then define

$$\Phi(x,\, y) = \int_\Gamma \mu\, a\, dx + \mu\, b\, dy, \tag{3.6}$$

where Γ is any curve (in the set) connecting Q to $(x,\, y)$. The value of this integral does not depend on the curve Γ, as follows from Green's theorem and (3.1) — i.e.: $\oint_\Lambda \mu\, a\, dx + \mu\, b\, dy = 0$ for any closed curve Λ in the set.[6] Hence (3.6) does define a function — which (obviously) satisfies (3.5).

Given (3.5), equation (3.2) yields

$$0 = \mu\, a\, dx + \mu\, b\, dy = \Phi_x\, dx + \Phi_y\, dy = d\Phi \quad \Longleftrightarrow \quad \Phi = \text{constant}. \tag{3.7}$$

Of course, for (3.7) to be of any use, μ must be non-trivial: $\mu = 0$ always works, but it also leads to the useless function $\Phi(x,\, y) \equiv \text{constant}$.

Vice-versa, if μ is an integrating factor, there is some function Φ such that

$$\mu\, a\, dx + \mu\, b\, dy = d\Phi = \Phi_x\, dx + \Phi_y\, dy. \tag{3.8}$$

Hence (3.5) applies, from which (3.1) follows.

Part II. With $a = 3\,x\,y + 2\,y^2$, and $b = 3\,x\,y + 2\,x^2$, equation (3.1) takes the form

$$a\,\mu_y - b\,\mu_x = (b_x - a_y)\,\mu = (x - y)\,\mu. \tag{3.9}$$

In terms of the coordinates $u = x + y$ and $v = x\,y$, a little bit of algebra reduces this equation to

$$(y - x)\,(2\,u\,\mu_u - v\,\mu_v + \mu) = 0. \tag{3.10}$$

That is, **for $x \neq y$ we have**

$$2\,u\,\mu_u - v\,\mu_v + \mu = 0. \tag{3.11}$$

[6] **Here is where having a set without holes matters:** if there are holes, then $\oint_\Lambda \mu\, a\, dx + \mu\, b\, dy = 0$ is not guaranteed, and (3.6) cannot be used to define a function.

We **only need one non-trivial solution for this equation to find an integrating factor.** Nevertheless, next we find **all** the solutions, using characteristics.

The characteristic equations for (3.11) can be written in the form

$$\frac{du}{2\,u} = -\frac{dv}{v} = -\frac{d\mu}{\mu} = ds, \tag{3.12}$$

where s is a parameter along each characteristic. From the first equality it follows that $u\,v^2 = \zeta$, where ζ is a constant on each characteristic — which we use as a label for the characteristic curve. From the second equality it follows that $\mu/v = f$, where f is also a constant along each characteristic — hence $f = f(\zeta)$ must be some function of the characteristic label. Thus the general solution to (3.11) has the form

$$\mu = v\,f(u\,v^2) = x\,y\,f\left((x+y)\,x^2\,y^2\right), \tag{3.13}$$

where f is some arbitrary function (with, at least, one derivative).

Remark 3.2 *The formula in (3.13) follows from solving (3.11), which is equivalent to (3.9) only for $x < y$, or $x > y$. Hence, at this stage, the only thing that we can say about the general solution to (3.9) is that it has the form*

$$\mu = x\,y\,f\left((x+y)\,x^2\,y^2\right) \quad \textit{for } x > y, \quad \textit{and} \quad \mu = x\,y\,g\left((x+y)\,x^2\,y^2\right) \quad \textit{for } x < y,$$

for some arbitrary function f and g. However, evaluating along $x = y$ yields

$$x^2\,f\left(2\,x^5\right) = x^2\,g\left(2\,x^5\right).$$

Thus, we conclude that:

All the solutions to (3.9) satisfy $\mu(x,\,y) = \mu(y,\,x)$, and have the form in (3.13)

— *with the same function f for all x and y.* This is rather interesting, since (generally) the fact that a p.d.e. has a symmetry **does not** imply that all the solutions have it too. *For example: the heat equation $T_t = T_{x\,x}$ is invariant under $x \leftrightarrow -x$, but $T = 2 + \sin(x)\,e^{-t}$ is a solution that is not invariant under $x \leftrightarrow -x$.*

We now take the simplest of the solutions in (3.13), $\mu = v = x\,y$, and plug it into equation (3.5). This yields

$$\Phi_x = a\,x\,y = 3\,x^2\,y^2 + 2\,x\,y^3 \quad \text{and} \quad \Phi_y = b\,x\,y = 3\,x^2\,y^2 + 2\,x^3\,y. \tag{3.14}$$

Thus $\boldsymbol{\Phi = x^3\,y^2 + x^2\,y^3 + \Phi_0}$ — where Φ_0 is a constant. Hence, from (3.7), it follows that: **For the case $\boldsymbol{a = 3\,x\,y + 2\,y^2}$ and $\boldsymbol{b = 3\,x\,y + 2\,x^2}$, (3.2) can be integrated to**

$$x^3\,y^2 + x^2\,y^3 = u\,v^2 = \text{constant}. \tag{3.15}$$

Part III. If μ is a non-trivial integrating factor for (3.4), then

$$\mu\,a\,dx + \mu\,b\,dy + \mu\,c\,dz = d\Phi, \tag{3.16}$$

for some function $\Phi = \Phi(x,\,y,\,z)$. This is equivalent to

$$\mu\,a = \Phi_x, \quad \mu\,b = \Phi_y, \quad \text{and} \quad \mu\,c = \Phi_z. \tag{3.17}$$

However, this then implies the equations

$$(\mu\,a)_y = (\mu\,b)_x, \quad (\mu\,c)_x = (\mu\,a)_z, \quad \text{and} \quad (\mu\,b)_z = (\mu\,c)_y. \tag{3.18}$$

Thus, we get three equations for a single unknown μ. This is an over-determined system that (generally) has only one solution: $\mu = 0$. In fact, in (3.18), multiply the first equation by c, the second equation by b, the third equation by a, add, and use the fact that we want $\mu \neq 0$. This then yields

$$0 = a\,(b_z - c_y) + b\,(c_x - a_z) + c\,(a_y - b_x). \tag{3.19}$$

Equivalently

$$0 = \mathbf{w}\cdot(\nabla\times\mathbf{w})\,, \quad \text{where} \quad \mathbf{w} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}. \tag{3.20}$$

Hence, if a, b, and c do not satisfy equation (3.20), (3.4) has no integrating factor.

TASK LEFT TO THE READER: *Show that, at least locally, (3.20) is sufficient to guarantee that (3.18) has a non-trivial solution.*

Hint 3.2 *If $\mathbf{w} \equiv 0$, then any μ solves (3.18). Hence, in any sufficiently small cube, we can assume that one of the components of $\mathbf{w}$ is never zero. Thus, without loss off generality, assume that $a \geq \delta > 0$ in the cube $-\epsilon < x,\,y,\,z < \epsilon$, where δ and $\epsilon > 0$ are constants. Then:* **I.** *Use the first equation in (3.18) to construct $\mu_0 = \mu_0(x,\,y)$ for $-\epsilon < x,\,y < \epsilon$.* **II.** *Use the second equation in*

(3.18), with $\mu(x,\,y,\,0) = \mu_0(x,\,y)$, to define μ in a neighborhood in $\mathcal{R}^{\ni}$ of the square $-\epsilon < x,\, y < \epsilon$.
III. *Show that the function μ that you just constructed solves (3.18). To do this:*
Define $\boldsymbol{\phi} = \boldsymbol{(\mu\, a)_y - (\mu\, b)_x}$ and $\boldsymbol{\psi} = \boldsymbol{(\mu\, c)_y - (\mu\, b)_z}$. Then: (i) Use (3.19) to find an algebraic relationship between ϕ and ψ. (ii) Use that μ satisfies the middle equation in (3.18), to derive a p.d.e. that ϕ satisfies. (iii) By construction $\phi = 0$ for $z = 0$. Use the p.d.e. in (ii) to conclude that $\phi \equiv 0$. (iv) Use (i) and (iii), conclude that $\psi \equiv 0$. Since μ satisfies the the middle equation in (3.18) by construction, this ends the proof.

Note: point out where the assumption $a > 0$ comes into play in your arguments.

Remark 3.3 *Of course, once a μ satisfying (3.18) is obtained, a Φ satisfying (3.17) is given by*

$$\Phi(x,\,y,\,z) = \int_\Gamma \mu\, a\, dx + \mu\, b\, dy, \tag{3.21}$$

where Γ is any curve connecting some fixed point Q with $(x,\,y,\,z)$. **Why is it that the integral in (3.21) depends ONLY on the endpoints of the curve Γ?**

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4 Quasi-Linear 1st order PDE (problem 03).

4.1 Statement: Quasi-Linear 1st order PDE (problem 03).

Simple waves in Gas Dynamics. Under the isentropic flow assumption, the Euler equations of Gas Dynamics in one dimension can be written in the form

$$\begin{aligned} \rho_t + (\rho\, u)_x &= 0, \quad \text{conservation of mass,} && (4.1)\\ (\rho\, u)_t + (\rho\, u^2 + p)_x &= 0, \quad \text{conservation of momentum,} && (4.2) \end{aligned}$$

where ρ is the mass density, u is the flow velocity, and p is the pressure. In addition, an **equation of state** must be provided:

$$p = P(\rho), \quad \text{satisfying } \frac{dP}{d\rho} = c^2 > 0, \tag{4.3}$$

where $\boldsymbol{c = c(\rho) > 0}$ has the dimensions of a velocity (it is the sound speed).

This is a system of two equations for two unknowns. Interestingly, the system has solutions that depend on a single unknown function. Namely, solutions of the form

$$\rho = R(\psi) \quad \text{and} \quad u = U(\psi), \tag{4.4}$$

where R and U are functions of the single argument ψ, and $\psi = \psi(x,\, t)$ satisfies a **scalar** quasi-linear equation of the form:

$$\psi_t + \lambda(\psi)\,\psi_x = 0. \tag{4.5}$$

Your **TASK** is to **find these solutions; that is: find $\boldsymbol{R}$, $\boldsymbol{U}$, and $\boldsymbol{\lambda}$.**

Hints. *(i) Characterize the functions R and U as solutions of an ode — do not seek explicit expressions. (ii) After you substitute (4.4) into the equations, cast the system into the form $A\,Y = 0$, where A is a 2×2 matrix. Then nontrivial solutions will exist if and only if* $\det(A) = 0$.

4.2 Answer: Quasi-Linear 1st order PDE (problem 03).

Substituting (4.4) into the equations yields

$$\begin{aligned} \psi_t\, R' + \psi_x\, (R\,U)' &= 0, &(4.6)\\ \psi_t\, U' + \psi_x \left(U\, U' + \frac{1}{R}\, c^2(R)\, R' \right) &= 0, &(4.7) \end{aligned}$$

where the primes denote derivatives with respect to ψ, and, instead of (4.2), we have used the equivalent form $\boldsymbol{u_t + u\, u_x + \frac{1}{\rho}\, p_x = 0}$. These equations can be re-cast into the form

$$\begin{bmatrix} \psi_t + U\,\psi_x & R\,\psi_x \\ (c^2/R)\,\psi_x & \psi_t + U\,\psi_x \end{bmatrix} \begin{bmatrix} R' \\ U' \end{bmatrix} = 0, \tag{4.8}$$

which has a non-trivial solution if and only if the determinant of the coefficient matrix vanishes. Namely $(\psi_t + U\,\psi_x)^2 - c^2\,\psi_x^2 = 0$. Thus, **there are two possible solutions**

$$\psi_t + (U \pm c(R))\,\psi_x = 0. \tag{4.9}$$

Substituting this into (4.8) reduces the two equations to just one, namely

$$\mp c(R)\, R' + R\, U' = 0 \quad \Longleftrightarrow \quad \mp c(R)\, dR + R\, dU = 0. \tag{4.10}$$

Thus, if $R = R(\psi)$ and $U = U(\psi)$ are functions related by this differential relation, and ψ satisfies (4.9), then (4.4) is a solution to the isentropic Euler equations of Gas Dynamics. Note that: **I.**

$\boldsymbol{\lambda = U \pm c(R)}$ is a function of ψ, via the dependence of U and R on ψ. **II.** The characteristic speed λ, at which information moves, for (4.9), is the flow speed plus or minus the sound speed c. In Gas Dynamics, c is the speed at which information moves relative to the fluid particles.

Example 4.1 *Consider an ideal gas, for which (in a-dimensional units)* $p = \dfrac{1}{\gamma}\rho^\gamma$*, with* $\gamma > 1$*. Then* $c = \rho^{(\gamma-1)/2}$*, and (4.10) can be integrated:*

$$\mp\frac{2}{\gamma-1}\,R^{(\gamma-1)/2} + U = U_0, \tag{4.11}$$

where U_0 *is a constant. Thus, we can take*

$$R = \psi, \quad U = U_0 \pm \frac{2}{\gamma-1}\,\psi^{(\gamma-1)/2}, \quad \textit{and} \quad \lambda = U_0 \pm \frac{\gamma+1}{\gamma-1}\,\psi^{(\gamma-1)/2}. \tag{4.12}$$

5 P.D.E. numerical integration (problem 01).

5.1 Statement: P.D.E. numerical integration (problem 01).

Consider the linear p.d.e.

$$u_t + (c(x)\,u)_x = 0, \quad \text{where} \quad c = 1 + \frac{1}{4}\,cos(x), \tag{5.1}$$

and $\boldsymbol{u}$ **is periodic of period** $\boldsymbol{2\,\pi}$ — i.e.: $\boldsymbol{u(x + 2\,\pi,\, t) = u(x,\, t)}$. Assume now that you are asked to calculate the solution of this p.d.e. for $\boldsymbol{0 \leq t \leq T = 6}$, with **initial condition** given by

$$u(x,\,0) = u_0(x) = \exp(-x^2) \quad \text{for} \quad -\pi \leq x \leq \pi, \tag{5.2}$$

extended periodically outside the interval $[-\pi,\,\pi]$.

You can extract a lot of information from the solution by characteristics of the problem above, but actual numerical values are not easy to access from it. For this, the best thing to do is to integrate the problem numerically. Here we will consider a few naive numerical algorithms for this purpose.

First, introduce a numerical grid, as follows:

$$x_n = -\pi + n\,h \quad \text{for} \quad 1 \leq n \leq N, \qquad \text{and} \qquad t_m = m\,k \quad \text{for} \quad 0 \leq n \leq M, \tag{5.3}$$

where N and M are "large" integers, $\boldsymbol{h = 2\,\pi/N}$, and $\boldsymbol{k = T/M}$. Let $\boldsymbol{u_n^m}$ **be the numerical solution's grid point values.** The expectation is that these values will be related to the exact solution $u(x,\, t)$ by $\boldsymbol{u_n^m = u(x_n,\, t_m) +}$ **small error**, with the error vanishing as N and M grow.

Next, consider the following numerical discretizations of the problem, which arise upon replacing the derivatives in the equation by finite differences that approximate them up to errors of some positive order in h or k. **In all cases the formulas apply for** $\boldsymbol{m \geq 0}$**, with** $\boldsymbol{u_n^0 = u_0(x_n)}$**.**

A. $\displaystyle \boldsymbol{0 = \frac{1}{k}\,(u_n^{m+1} - u_n^m) + \frac{1}{h}\,\Big(c(x_n)\,u_n^m - c(x_{n-1})\,u_{n-1}^m\Big)}.$

B. $\displaystyle \boldsymbol{0 = \frac{1}{k}\,(u_n^{m+1} - u_n^m) + \frac{1}{h}\,\Big(c(x_{n+1})\,u_{n+1}^m - c(x_n)\,u_n^m\Big)}.$

C. $\displaystyle \boldsymbol{0 = \frac{1}{k}\,(u_n^{m+1} - u_n^m) + \frac{1}{2\,h}\,\Big(c(x_{n+1})\,u_{n+1}^m - c(x_{n-1})\,u_{n-1}^m\Big)}.$

In all cases, once u_n^m is known for some m and all n, u_n^{m+1} can be explicitly computed, for all n. **Note:** when a formula above in **A**, **B** or **C**, calls for a value u_n^m outside the range $1 \leq n \leq N$, the periodic boundary conditions, which translate into $\boldsymbol{u_{n+N}^m = u_n^m}$, must be used.

The **tasks in this problem are:**

1. **Causality and numerics.** Using arguments based **solely** on how the information propagates in the exact solution (characteristics), versus how it propagates in the numerical schemes above, **argue that:** *(1.1) One of the schemes above cannot possibly work. (1.2) A necessary condition for the other two to work is that a restriction of the form $k \leq \lambda\, h$ be imposed on the time step* — where $\lambda > 0$ is a constant that depends on $c = c(x)$.

2. Implement the schemes and try them out with various space resolutions,[7] and a corresponding time resolution. Do you see convergence? Do the results agree with your analysis in item **1**? **Report what you see, and illustrate it with plots** — a **few** well selected plots should be enough!

5.2 Answer: P.D.E. numerical integration (problem 01).

Causality. The key question we have to address first, for both the p.d.e. as well as its discretized

[7] N in the range $20 \leq N \leq 200$ should be more than enough to see what happens.

versions, is: **what are the domains of dependence?** Namely:

Given some arbitrary point $P = (x_0,\, t_0)$ in space-time, what is the region with the property that changes there affect the value of the solution at P? (5.4)

For us here, this is important because **what happens outside the domain of dependence of a point P_0, has NO EFFECT at all on the value of the solution there.** Hence

A necessary condition for the solution computed by a numerical algorithm to converge to the solution of a p.d.e. (as the numerical grid is refined), is: The numerical domain of dependence for any point P should include (as the numerical grid is refined) the p.d.e. domain of dependence for P. (5.5)

Proof: how can the numerical algorithm get the correct values for the solution of a p.d.e., without using the data that determines the p.d.e. solution? **Note:** restrictions on numerical algorithms that arise in this fashion are called **C.F.L. conditions** (C.F.L. = Courant, Friedrichs, and Lewy).

So, given an arbitrary point $P = (x_0,\, t_0)$, what are the domains of dependence relevant here?

1. **For equation (5.1)** the domain of dependence is given by the characteristic through P. Namely, by the curve determined by the o.d.e. problem:

$$\frac{dx}{dt} = c(x), \quad \text{for } t < t_0, \quad \text{with "initial condition" } x(t_0) = x_0.$$

Hence, let $\boldsymbol{c_{\max} = \max_{|x|\leq\pi} c(x) = 1.25,}$ and $\boldsymbol{c_{\min} = \min_{|x|\leq\pi} c(x) = 0.75.}$ Then, **the p.d.e. domain of dependence is included within the wedge:**

$$x_0 + c_{\max}\,(t - t_0) \leq x \leq x_0 + c_{\min}\,(t - t_0), \quad \text{with} \quad t < t_0. \tag{5.6}$$

Note that, for a "generic" $c = c(x)$, and arbitrary P, this wedge is the best one can do.

2. **For the scheme in item A** the domain of dependence is given by

$$x_0 + \frac{h}{k}\,(t - t_0) \leq x \leq x_0, \quad \text{with} \quad t < t_0. \tag{5.7}$$

Note that, for any fixed h and k, the domain is a discrete set of points. However, as the grid is refined, the whole (5.7) wedge fills up. The same applies to the wedges in (5.8 – 5.9).

3. **For the scheme in item B** the domain of dependence is given by

$$x_0 \leq x \leq x_0 - \frac{h}{k}\,(t - t_0), \quad \text{with} \quad t < t_0. \tag{5.8}$$

4. **For the scheme in item C** the domain of dependence is given by

$$x_0 + \frac{h}{k}(t - t_0) \leq x \leq x_0 - \frac{h}{k}(t - t_0), \quad \text{with} \quad t < t_0. \tag{5.9}$$

Thus, **in order for (5.5) to apply, we need:**

— **Scheme in item A.** The wedge in (5.6) will be included within the wedge in (5.7) provided that $h/k \geq c_{\max} = 1.25$. Equivalently:

$$k \leq \frac{1}{c_{\max}} h = \frac{1}{1.25} h \quad \Longleftrightarrow \quad M \geq \frac{1.25}{2\pi} T N. \tag{5.10}$$

— **Scheme in item B.** The wedge in (5.6) **cannot** be included within the wedge in (5.8) for any choice of $h > 0$ and $k > 0$. **This scheme cannot/will not work.**

— **Scheme in item C.** The wedge in (5.6) will be included within the wedge in (5.9) provided that $h/k \geq c_{\max} = 1.25$. Equivalently: **(5.10) must apply.**

Condition (5.10) is necessary only. It DOES NOT guarantee that either scheme (A or C) works, nor does it tell us how the scheme in item B fails. To check what happens, I implemented the schemes. The results (which **do not contradict the theoretical conclusions** above) follow.

Implementation of the scheme in item A.

As long as the calculation is performed satisfying the constraint in (5.10) — C.F.L. condition — the scheme appears to perform properly, with convergence as the grid is refined — see figure 5.1.

Remark 5.1 **IMPORTANT NOTE.** *Since I do not have an exact solution to compare the numerical results with, I cannot (solely from the numerical calculation) be sure that the observed convergence is to the actual solution* [Unfortunately, it is not too hard to produce very reasonable looking numerical algorithms that converge to the wrong answer, this even for rather simple p.d.e.'s]. *Fortunately, there are many tests one can do (using solely the numerical results) to increase the confidence level beyond "it looks right" — stuff not covered in this course, but you can learn it by taking a course such as 18.336. At any rate:* in the specific case of the scheme in item A, one can prove that: as $h \to 0$, with k restricted by (5.10), the numerical solution converges to the solution to (5.1 – 5.2).

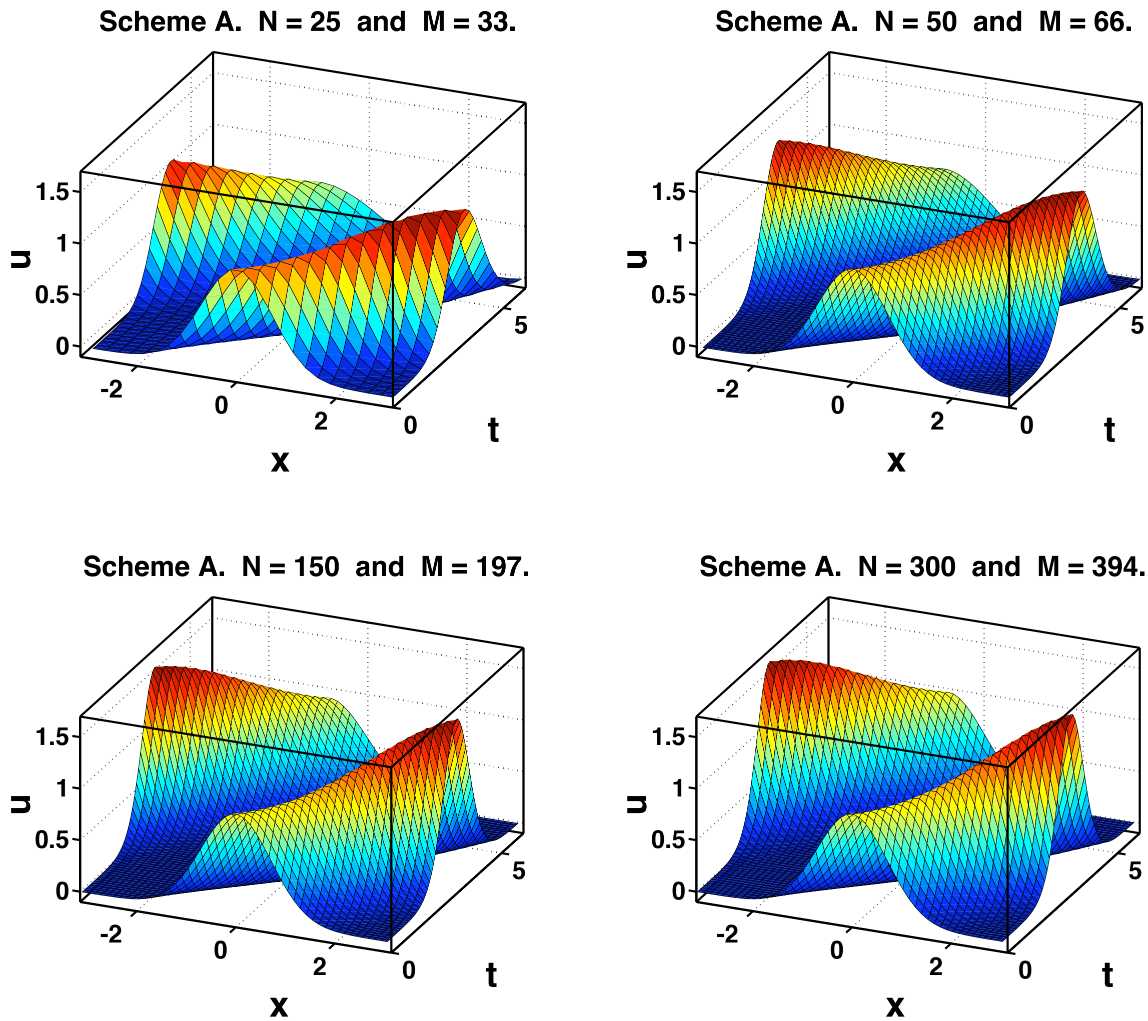

Figure 5.1: Numerical calculations with the scheme in item **A**, to solve the problem in (5.1 – 5.2). From left to write, and top to bottom, plots of the numerical solutions as the numerical grid is refined, enforcing the C.F.L. condition in (5.10) — specifically, take M as close to $M = 1.1\,(1.25\,T\,N)/(2\,\pi)$ as allowed by the fact that M is an integer. Convergence appears to be occurring.

Implementation of the scheme in item B.

When implementing this scheme, grid scale oscillations (dominant wavelength is $2\,h$), which grow exponentially with time (and reach huge amplitudes very quickly) appear. In fact, the growth rate of the oscillations increases as the grid is refined, producing catastrophic results. This scheme is not content with simply failing to work, it fails spectacularly (which is nice: makes it easy to tell it is not working). This is illustrated by the pictures in figure 5.2.

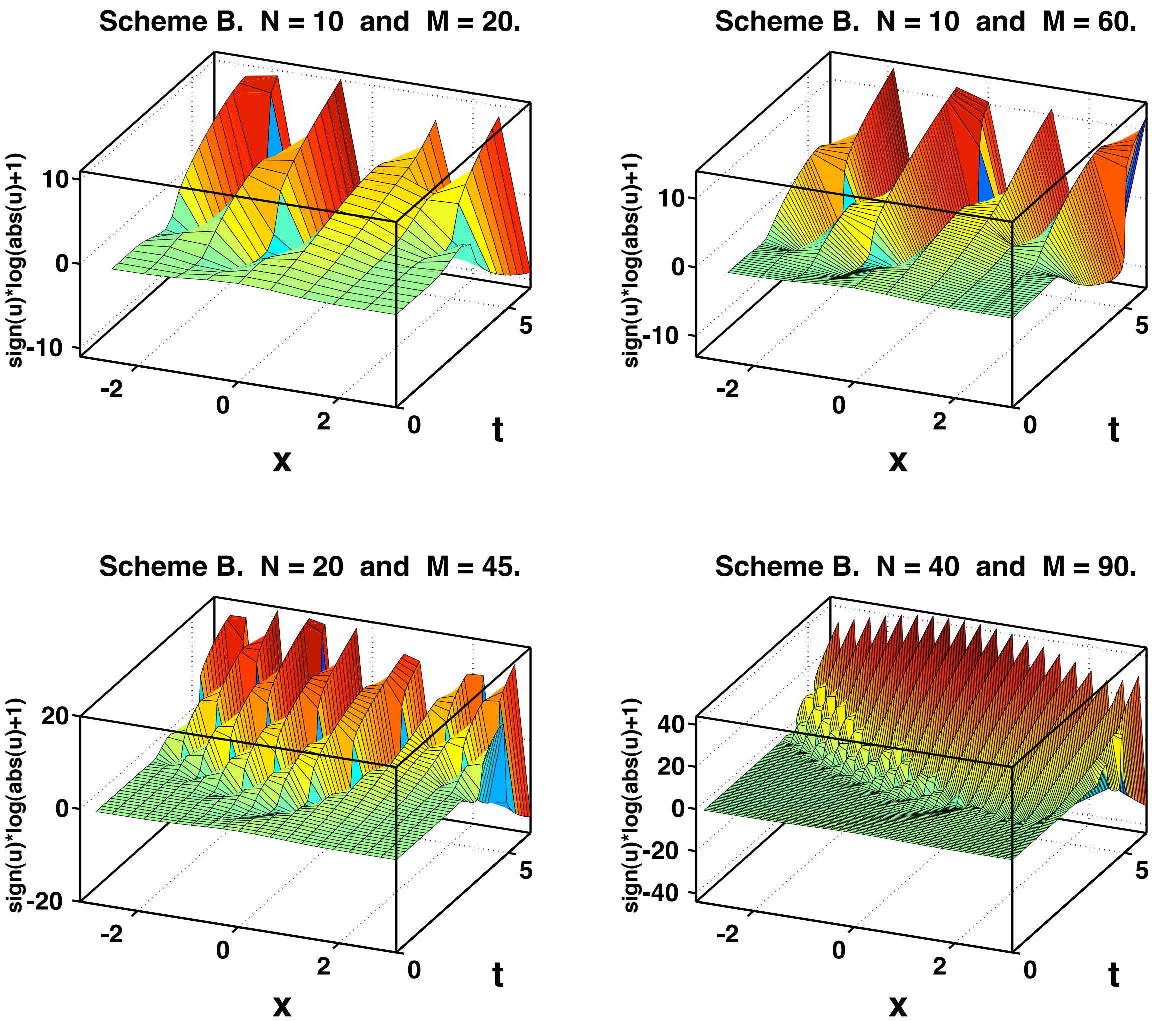

Figure 5.2: Numerical calculations with the scheme in item **B**, to solve the problem in (5.1 – 5.2). Left to write, and top to bottom: numerical solution, as the numerical grid is refined — with M substantially above the value required by (5.10). Taking M even larger does not help, as illustrated by the top row. The amplitude of the spurious oscillations generated is very large, even for moderate values of N. Hence, we do not plot the numerical solution directly: The plots are for $\mathbf{sign}(\boldsymbol{u})\ \mathbf{ln}(\mathbf{1} + |\boldsymbol{u}|)$!

Implementation of the scheme in item C.

This scheme does not work either, though it does not fail as spectacularly as the scheme in item **B** — the grid scale oscillations have (comparatively) slower growth ratios. Nevertheless: grid

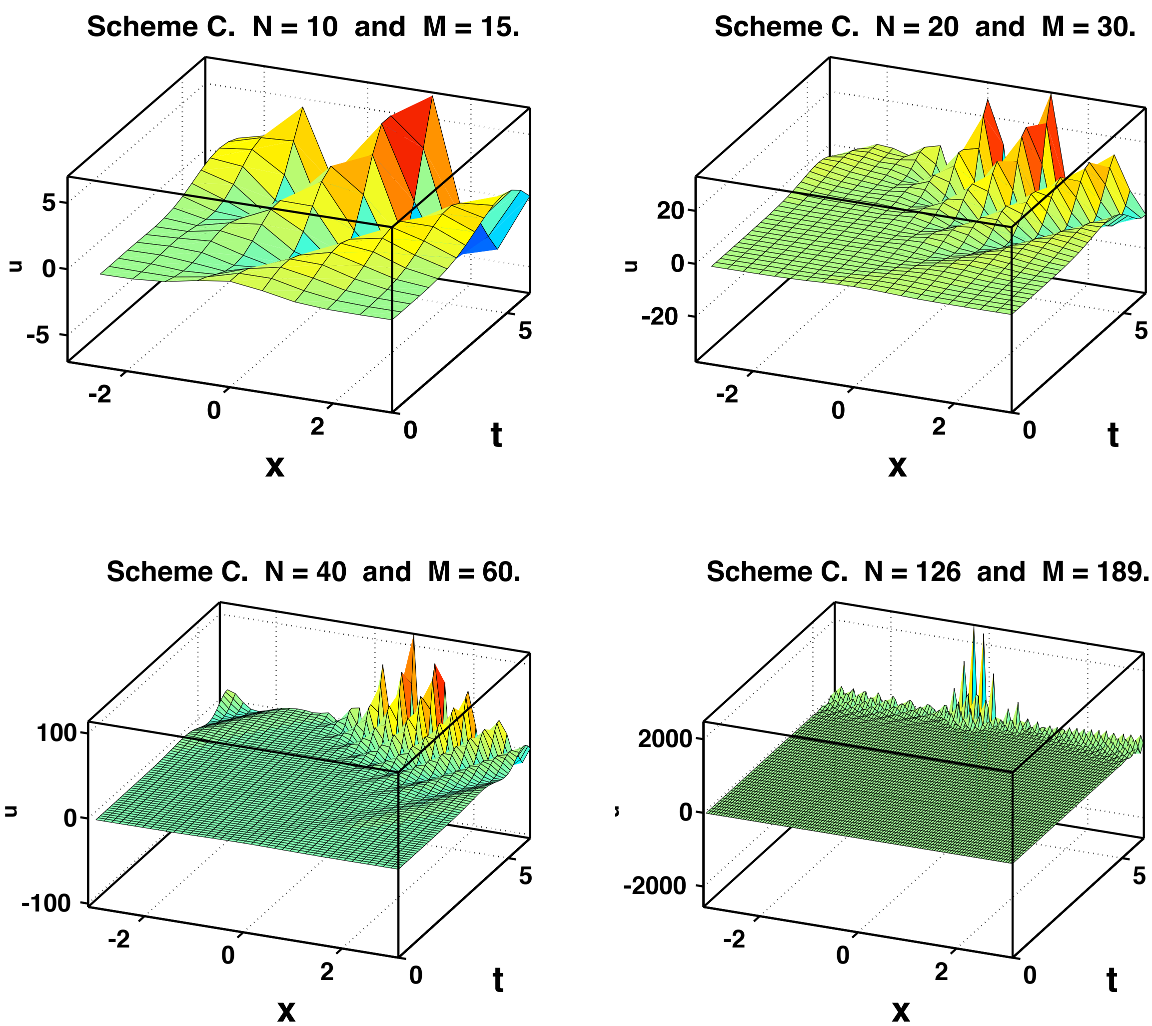

Figure 5.3: Numerical calculations with the scheme in item **C**, to solve the problem in (5.1 – 5.2). Left to write, and top to bottom: numerical solution, as the numerical grid is refined — with M substantially above the value required by (5.10). Taking M even larger does not help.

scale oscillations are generated, which grow exponentially with time. Furthermore, the growth rate of these oscillations becomes larger as the grid is refined. See figure 5.3.

Remark 5.2 *If the scheme in item* **A** *is used under conditions that violate the C.F.L. condition in (5.10), then behavior similar to the one observed for the schemes in items* **B** *and* **C** *appears. Namely: grid scale oscillations that grow exponentially, with growth rates becoming larger as* $h \to 0$.

Note: if the C.F.L. violation is "small", then the growth rate of the oscillations is small when N

is moderate. Hence, the grid scale oscillations will be hard to see, unless N is large enough, or the interval of computation (i.e.: T) is long enough.

Remark 5.3 *Not only is it possible to prove that the scheme in item* **A** *converges if the C.F.L. condition is enforced — see remark* ***5.1***. *It is also possible to explain, theoretically, why and when grid scale oscillations will appear, and to estimate their growth rate.*

THE END.