18.306 Advanced Partial Differential Equations with Applications
Fall 2009

Answers to Problem Set Number 04 for 18.306 — MIT (Fall 2009)

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1 Eikonal equation (problem 01).

1.1 Statement: Eikonal equation (problem 01).

Consider the Eikonal equation (for the wave equation in 2-D) in a context where the wave speed is a constant (homogeneous media), so that we can set (upon non-dimensionalization) $c = 1$. Then

$$\phi_x^2 + \phi_y^2 = 1. \tag{1.1}$$

Consider now the situation where the wave-front $\phi = 0$ is a parabola. Specifically:

$$\phi = 0 \quad \text{on} \quad y = x^2, \tag{1.2}$$

with propagation direction towards y increasing. For this problem, **this is what you should do:**

1. Find the family of all the rays (characteristics) for $t > 0$. The easiest way is to describe it is parametrically: $x = x(s,\, t)$ and $y = y(s,\, t)$, where $x(s,\, 0) = s$, $y(s,\, 0) = s^2$, and t is time of travel along the ray (for the wave-fronts) starting from the initial wave-front — i.e.: $\phi = t$.

2. Find the caustic. The caustic is the envelope of the family of rays = the locus of the intersections of infinitely close neighbors in the family of rays[1] = a curve such that each point in it belongs to one of the rays, and it is tangent to the ray there.[2] In this case of constant wave speed, where the rays are straight lines, the caustic is also the locus of the centers of curvature of the wave-fronts (all the wave-fronts have the same set of centers of curvature).

From the parametric description of the family of rays in item **1**, you should be able to obtain the caustic parametrically in terms of s. However, you should also be able to find a very simple formula — of the form $(y - y_a)^\alpha = \text{const.}\,(x - x_a)^2$ — for the caustic. Do so.

3. Do a sketch of the wave-front $\phi = 0$, and of the caustic. Indicate the region of the plane where the rays cross and give rise to multiple values in the solution to the equation.

4. The earliest time at which a ray crossing occurs corresponds to the singular point in the caustic (the arête). Find the position of the arête in space, the ray, and the time (or wave-front, as

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[1] $(x_*,\, y_*) \in$ caustic $\iff$ $x_* = x(s,\, t) = x(s + ds,\, t + dt)$ and $y_* = y(s,\, t) = y(s + ds,\, t + dt)$, for some s and t.

[2] $(x_*,\, y_*) \in$ caustic $\iff$ for some s and t: $x_* = x(s,\, t)$, $y_* = y(s,\, t)$, and $(x_t(s,\, t),\, y_t(s,\, t))$ is tangent to the caustic at $(x_*,\, y_*)$.

$\phi = t$) it corresponds to. Let these parameters be x_a, y_a, s_a, and t_a. Explicitly show that t_a is the earliest time at which a crossing of rays occurs.

5. Add to the sketch in item **3** the wave-front $\phi = t_a$. This wave-front is singular at the arête; describe the nature of this singularity. In particular, show that the wave-front satisfies (at leading order) a formula of the form $(y - y_a) \sim \text{const.}\,(x - x_a)^{\mu}$ near the arête.

1.2 Answer: Eikonal equation (problem 01).

The characteristic equations for the Eikonal equation (1.1) are given by

$$\frac{d\vec{r}}{dt} = \vec{p}, \quad \frac{d\vec{p}}{dt} = 0, \quad \text{and} \quad \frac{d\phi}{dt} = 1, \tag{1.3}$$

where $\vec{r} = (x,\, y)$, $\vec{p} = \nabla\phi$, and t is the travel time along the rays for the wave front. These must be given the initial ($t = 0$) values

$$\vec{r} = (s,\, s^2), \quad \vec{p} = \hat{n} = \frac{1}{\sqrt{1+4s^2}}\,(-2\,s,\, 1), \quad \text{and} \quad \phi = 0, \tag{1.4}$$

where $-\infty < s < \infty$ parameterizes the initial wave front — note that $\hat{n}$ is the unit normal to the initial wave front, pointing towards the direction of propagation.

The solution to (1.3–1.4) is given by

$$x = s - \frac{2\,s}{\sqrt{1+4\,s^2}}\,t = s - 2\,s\,\tau, \quad y = s^2 + \frac{1}{\sqrt{1+4\,s^2}}\,t = s^2 + \tau, \tag{1.5}$$

$\vec{p} = \hat{n}$ and $\phi = t$, where $\tau = t/\sqrt{1+4\,s^2}$. Then

1. The family of rays is given by (1.5), with s labeling each ray, and t (or τ) parameterizing the ray. Alternatively, substituting $\tau = y - s^2$ into the formula for x, we obtain the alternative description

$$x + 2\,s\,y - s - 2\,s^3 = 0. \tag{1.6}$$

2. In order to find the envelope of the family of rays (the caustic), we solve the equations

$$x(s,\, \tau) = x(s + ds,\, \tau + d\tau) \quad \text{and} \quad y(s,\, \tau) = y(s + ds,\, \tau + d\tau). \tag{1.7}$$

Equivalently

$$\begin{pmatrix} x_s & x_\tau \\ y_s & y_\tau \end{pmatrix} \begin{pmatrix} ds \\ d\tau \end{pmatrix} = 0. \tag{1.8}$$

The determinant of the matrix must vanish, thus:

$$x_s\,y_\tau - x_\tau\,y_s = (1-2\,\tau)\,(1) - (-2\,s)\,2\,s = 1 + 4\,s^2 - 2\,\tau. \tag{1.9}$$

It follows that (along the caustic) $\tau = \dfrac{1+4\,s^2}{2}$ — equivalently $t = \dfrac{(1+4\,s^2)^{3/2}}{2}$. Substituting this into (1.5), we find that the caustic is given by

$$x = -4\,s^3 \quad \text{and} \quad y = \frac{1}{2} + 3\,s^2. \tag{1.10}$$

Eliminating s, this yields (for the caustic) the simple formula

$$\left(y - \frac{1}{2}\right)^3 = \frac{27}{16}\,x^2. \tag{1.11}$$

Alternatively, the family of rays is given by an equation of the form $f(x,\,y,\,s) = 0$ — see (1.6). Thus the caustic follows from (1.6) and $f(x,\,y,\,s) = f(x,\,y,\,s+ds) \Rightarrow f_s = 0$. With f as in (1.6), $f = f_s = 0$ leads again to (1.10–1.11).

Finally, the curvature along the initial front ($x = s$ and $y = s^2$) is given by

$$\kappa = \frac{x_s\,y_{ss} - x_{ss}\,y_s}{(x_s^2 + y_s^2)^{3/2}} = \frac{2}{(1+4\,s^2)^{3/2}}, \tag{1.12}$$

and the front is curved towards $y = +\infty$. Hence the locus of centers of curvature is given by

$$\vec{r} = (s,\,s^2) + \kappa^{-1}\,\hat{n}, \tag{1.13}$$

where $\hat{n}$ is as in (1.4). This last formula is the same as (1.10).

3. See figures 1.1 and 1.2.

4. From (1.9), $t = \dfrac{(1+4\,s^2)^{3/2}}{2}$ along the caustic — which is the boundary of the multiple values region for ϕ (see item **3**). This has a minimum for $s = 0$. Hence *the earliest time at which a ray crossing occurs is* $t_a = \dfrac{1}{2}$, *which happens along the ray with label* $s_a = 0$ — starting at the "nose" of the parabola that makes the initial wave front. *This crossing occurs at the position* $x_a = 0$ *and* $y_a = \dfrac{1}{2}$ — as follows from (1.10).

Notice that x_a and y_a correspond to the place where the caustic curve in (1.11) is singular, and has a cusp — the arête. *The* $2/3$ *cusp singularity at the arête is generic: any focusing smooth initial wave-front gives rise to a singularity of this type in the caustic.*

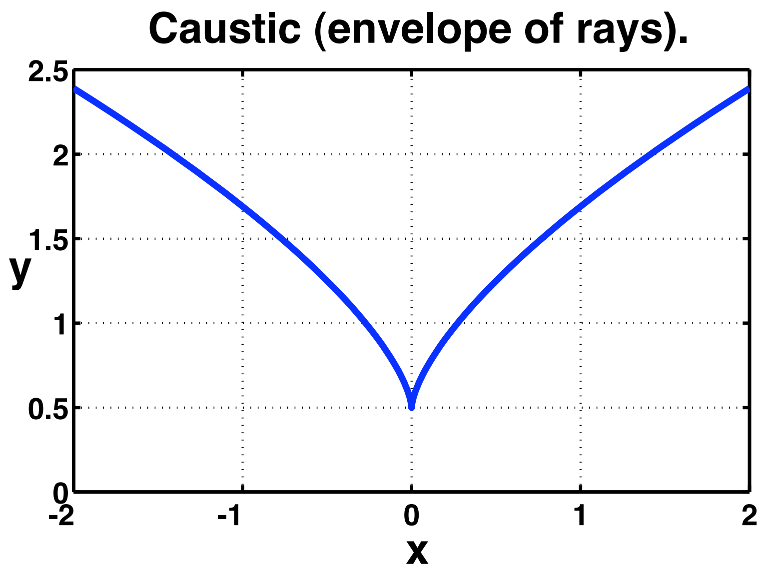

Figure 1.1: Caustic, as given by (1.11), for the problem in (1.1 – 1.2). The cusp is called the arête.

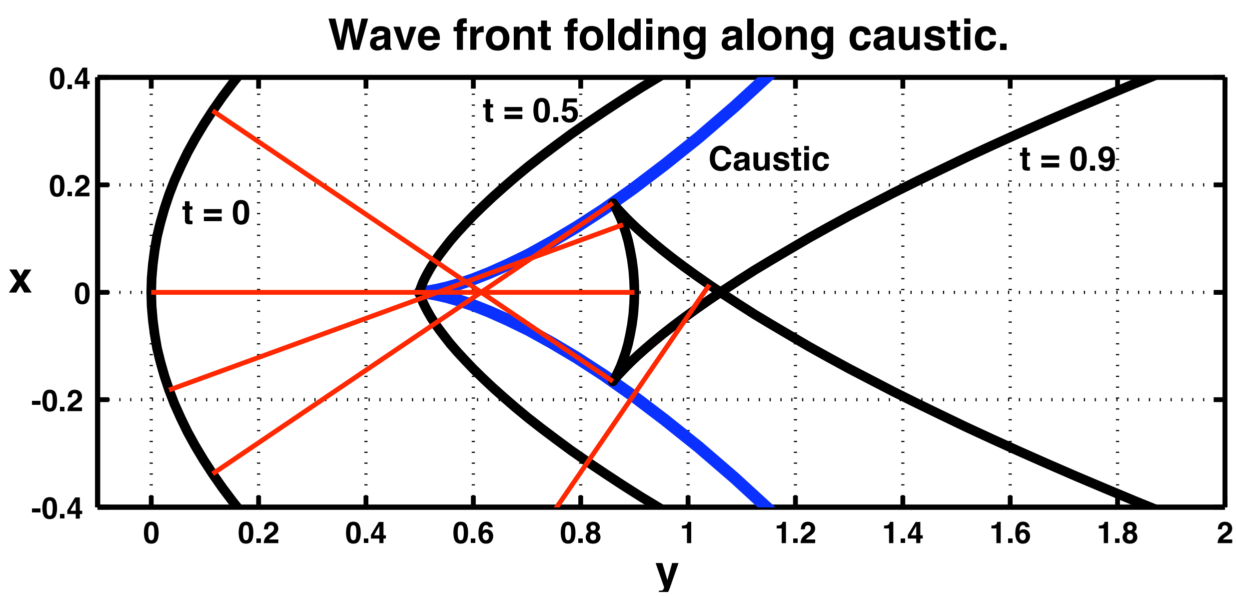

Figure 1.2: This picture shows the initial wave front, the wave front right at the time it is about to start folding (t $=$ 0.5), a typical wave front at a later time, the caustic, and a few rays. Multiple values (more than one ray through each point) occur in the area contained within the caustic curve.

Note: *a detailed proof that* t_a *is the earliest time for which a ray crossing occurs follows.*

Let $x = X(s,\, t)$ and $y = Y(s,\, t)$ be as in (1.5) — for each s fixed and t varying these equations give a ray, while for each t fixed and s varying these equations give a wavefront. Then

$$\frac{\partial X}{\partial s} = 1 - \frac{2}{(1+4\,s^2)^{3/2}}\,t \quad \text{and} \quad \frac{\partial Y}{\partial s} = 2\,s - \frac{4\,s}{(1+4\,s^2)^{3/2}}\,t = 2\,s\,\frac{\partial X}{\partial s}.$$

Now fix $t > 0$, and investigate the resulting wavefront. The following three cases occur:

Case $t < 1/2$. Then $\partial X/\partial s > 0$, so that the wave-front is a smooth curve, with a non-vanishing tangent vector everywhere. Along it x is a strictly increasing function of s, while y has a global minimum at $s = 0$ — y is strictly decreasing for $s < 0$, and strictly increasing for $s > 0$. Hence **no ray crossings occur.**

Case $t = 1/2$. Then $\partial X/\partial s > 0$ for $s \neq 0$ and $\partial X/\partial s = 0$ for $s = 0$. The wavefront is a smooth curve for $s \neq 0$, with a non-vanishing tangent vector. The wavefront is singular at $x = 0$ and $y = 1/2$ (corresponding to $s = 0$) — see equation (1.15). However, it is still true that along the wavefront x is a strictly increasing function of s, while y has a global minimum at $s = 0$ — y is strictly decreasing for $s < 0$, and strictly increasing for $s > 0$. Hence **the first ray crossing is happening, at $x = 0$ and $y = 1/2$.**

Case $t > 1/2$. Then a value $s_1 > 0$ exists such that $\partial X/\partial s > 0$ for $|s| > s_1$, $\partial X/\partial s < 0$ for $|s| < s_1$, and $\partial X/\partial s = 0$ for $|s| = s_1$. The wavefront is a smooth curve for $s \neq \pm s_1$, with a non-vanishing tangent vector. The wavefront is singular at $s = \pm s_1$ (corresponding to the points on the caustic). Along the wavefront x is a strictly increasing function of s for $|s| > s_1$, and a strictly decreasing function for $|s| < s_1$. The situation for y is a little more involved: y decreases for $s < -s_1$, increases for $-s_1 < s < 0$, decreases for $0 < s < s_1$, and increases for $s_1 < s$. Hence **plenty of ray crossings have occurred: the section of the wavefront corresponding to $|s| < s_1$ belongs to rays that have gone through a crossing.**

Note that there are plenty of ray crossings, but these all occur in the region "beyond" the caustic. An analytical proof of this fact is not easy, but a geometrical argument is intuitively easy, using the following facts: The rays are all straight lines tangent to the caustic somewhere — see remark **1.1** below. The caustic consists of two convex curves, joined at the arête, where they have a common tangent (the y-axis). By definition, the tangent lines to a convex curve are completely contained within a single side of the curve. The actual argument is left to the reader.

5. The $t = t_a = 1/2$ wave front is described, as follows from (1.5), by

$$x = s - \frac{s}{\sqrt{1+4\,s^2}}, \qquad y = s^2 + \frac{1}{2\sqrt{1+4\,s^2}}. \tag{1.14}$$

For s small this yields $x = 2\,s^3 + O(s^5)$ and $y = \dfrac{1}{2} + 3\,s^4 + O(s^6)$. Hence, *near the arête*

$$y - \frac{1}{2} = 3\,\left(\frac{x}{2}\right)^{4/3} + O(x^2). \tag{1.15}$$

Again, the wave front is singular at the arête, with a tangent there, but an infinite curvature — a 4/3 singularity. *This type of singularity is also generic for focusing smooth initial wave-fronts.*

Remark 1.1 *Note that the equations for the rays (and wavefronts) in (1.5) can be written in the form*

$$x = -4\,s^3 - \frac{2\,s}{\sqrt{1+4\,s^2}}\,(t - t_0) \quad \text{and} \quad y = \frac{1}{2} + 3\,s^2 + \frac{1}{\sqrt{1+4\,s^2}}\,(t - t_0), \tag{1.16}$$

where $t_0 = \dfrac{(1+4\,s^2)^{3/2}}{2}$. *Comparing with (1.10), we see that this formula indicates that* **the family of rays is the same as the family of straight lines tangent to the caustic,**[3] *with* t_0 *the time at which the ray is tangent to the caustic.*

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2 Eikonal equation (problem 03).

2.1 Statement: Eikonal equation (problem 03).

Consider the Eikonal equation (for the wave equation in 2-D) in a context where the wave speed is a constant (homogeneous media), so that one can set (upon non-dimensionalization) $c = 1$. Then

$$\phi_x^2 + \phi_y^2 = 1. \tag{2.1}$$

The *rays* for this equation are the lines defined by

$$\frac{dx}{dt} = \phi_x \quad \text{and} \quad \frac{dy}{dt} = \phi_y. \tag{2.2}$$

The characteristic form for (2.1) is then the o.d.e. system composed by the two equation in (2.2), plus the following three additional equations

$$\frac{d\phi_x}{dt} = 0, \quad \frac{d\phi_y}{dt} = 0, \quad \text{and} \quad \frac{d\phi}{dt} = 1. \tag{2.3}$$

[3] Not surprising, since the caustic is the envelope of the rays.

Your tasks are:

1. Find an equation for the evolution along the rays of the Hessian of ϕ. Namely, the matrix

$$M = \begin{bmatrix} \phi_{xx} & \phi_{xy} \\ \phi_{yx} & \phi_{yy} \end{bmatrix} \tag{2.4}$$

2. Solve the equation for M derived in item **1**, and write a formula giving the front curvature $\boldsymbol{\kappa = \phi_{xx} + \phi_{yy}}$ along each ray, as a function of t, and the front curvature $\boldsymbol{\kappa_0}$ on the ray at the wave front corresponding to $t = 0$ — recall that the wave fronts are given by $\phi(x,\, y) = t$. **Note:** the formula for κ involves t and κ_0 **only.**

Hints: (i) *Consider the second order derivatives (i.e. ∂^2_{xx}, ∂^2_{xy}, and ∂^2_{yy}) of equation (2.1). Then use that, for any $f = f(x,\, y)$, its derivative along the rays is given by $\dfrac{d}{dt} f = \phi_x\, f_x + \phi_y\, f_y$.* **(ii)** *To solve the equation for M, proceed as follows: Let M_0 be the value of M at $t = 0$. Define $W = (1 + M_0\, t)\, M$, and write the equation W satisfies. Using that $W = M_0$ at $t = 0$, you should now be able to solve this equation by inspection.* **(iii)** *Once you have solved the equation for M, use it to get the behavior of the eigenvalues of M — note that κ is the sum of the eigenvalues.* **(iv)** *Finally, inspect the gradient of equation (2.1). What does it tell you about the eigenvalues of M?*

2.2 Answer: Eikonal equation (problem 03).

From the Eikonal equation (2.1), it follows that

$$\begin{aligned} 0 &= \phi_x\, \phi_{xxx} + \phi_{xx}^2 + \phi_{yx}^2 + \phi_y\, \phi_{yxx}, \qquad (2.5) \\ 0 &= \phi_x\, \phi_{xxy} + \phi_{xy}\, \phi_{xx} + \phi_{xy}\, \phi_{yy} + \phi_y\, \phi_{yyx}, \qquad (2.6) \\ 0 &= \phi_x\, \phi_{xyy} + \phi_{xy}^2 + \phi_{yy}^2 + \phi_y\, \phi_{yyy}, \qquad (2.7) \end{aligned}$$

Thus

$$0 = \phi_{xx}^2 + \phi_{yx}^2 + \frac{d}{dt}\, \phi_{xx} = \phi_{xy}\, \phi_{xx} + \phi_{xy}\, \phi_{yy} + \frac{d}{dt}\, \phi_{xy} = \phi_{xy}^2 + \phi_{yy}^2 + \frac{d}{dt}\, \phi_{yy}. \tag{2.8}$$

Equivalently, in matrix form

$$\boldsymbol{0 = \frac{d}{dt} M + M^2 \quad \Longrightarrow \quad M = (I + M_0\, t)^{-1}\, M_0,} \tag{2.9}$$

where $\boldsymbol{M_0}$ is the value of M for $t = 0$, and $\boldsymbol{I}$ is the identity.

Proof: For any matrix $U = U(t)$, $\dfrac{d}{dt}U^{-1} = -U^{-1}\,\dot{U}\,U^{-1}$ — as follows from differentiating $U\,U^{-1} = I$.

Let now λ_1^0 and λ_2^0 be the eigenvalues of M_0, with corresponding eigenvectors[4] $\vec{v}_1$ and $\vec{v}_2$. Then, from (2.9), it follows that

$$M\,\vec{v}_j = \frac{\lambda_j^0}{1+\lambda_j^0\,t}\,\vec{v}_j \quad \text{for} \quad j = 1, 2. \tag{2.10}$$

Hence M **has the eigenvalues**

$$\boldsymbol{\lambda_j = \frac{\lambda_j^0}{1+\lambda_j^0\,t}} \quad \text{for} \quad \boldsymbol{j = 1, 2,} \tag{2.11}$$

with corresponding eigenvectors $\vec{v}_1$ and $\vec{v}_2$. Furthermore, by taking the gradient of equation (2.1), we obtain

$$M\,\nabla\phi = 0. \tag{2.12}$$

Hence: **one of the eigenvalues of M vanishes.** Since the curvature of the wave fronts is given by $\kappa = \text{Trace}(M)$, it follows that κ **is the other eigenvalue.** Therefore

$$\boldsymbol{\kappa = \frac{\kappa_0}{1+\kappa_0\,t}.} \tag{2.13}$$

In particular: if the initial wave front has negative curvature anywhere, then the curvature blows up at some finite positive time.

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3 Eikonal equation (problem 04).

3.1 Statement: Eikonal equation (problem 04).

Consider the Eikonal equation (for the wave equation in 2-D) in a context where the wave speed is a constant (homogeneous media), so that one can set (upon non-dimensionalization) $c = 1$. Then

$$\phi_x^2 + \phi_y^2 = 1. \tag{3.1}$$

[4]Since M_0 is a symmetric matrix, it has real eigenvalues, with an orthonormal set of eigenvectors.

The *rays* for this equation are the lines defined by
$$\frac{dx}{dt} = \phi_x \quad \text{and} \quad \frac{dy}{dt} = \phi_y. \tag{3.2}$$

Equation (3.1) accepts solutions with singularities along the rays. Specifically, solutions for which:

1. ϕ and $\nabla\,\phi$ are both continuous.

2. The second derivatives[5] of ϕ exist, but they fail to be continuous along some ray.

Exhibit a solution with these properties, valid in some region of the plane — say $x > 0$.

Hint: *A simple set of solutions for (3.1) is given by $\phi = \phi_0 + \sqrt{(x-x_0)^2 + (y-y_0)^2}$, where ϕ_0 is a constant and $P = (x_0,\, y_0)$ is a point in the plane — these solutions correspond to waves radiating from a single point in space. You can construct an example by gluing solutions of this type.*

3.2 Answer: Eikonal equation (problem 04).

Example 1. **Define ϕ for $x > 0$ by**

$$\phi = 2 + \sqrt{(x-2)^2 + y^2} \qquad \text{for } y \geq 0, \tag{3.3}$$
$$\phi = 1 + \sqrt{(x-1)^2 + y^2} \qquad \text{for } y \leq 0. \tag{3.4}$$

Clearly, ϕ is continuous. Furthermore:

$$\text{For } y \geq 0, \qquad \phi_x = \frac{x-2}{\sqrt{(x-2)^2+y^2}} \quad \text{and} \quad \phi_y = \frac{y}{\sqrt{(x-2)^2+y^2}}. \tag{3.5}$$

$$\text{For } y \leq 0, \qquad \phi_x = \frac{x-1}{\sqrt{(x-1)^2+y^2}} \quad \text{and} \quad \phi_y = \frac{y}{\sqrt{(x-1)^2+y^2}}. \tag{3.6}$$

Hence $\nabla\phi$ is also continuous. However

$$\text{For } y > 0, \qquad \phi_{yy} = \frac{1}{\sqrt{(x-2)^2+y^2}} - \frac{y^3}{((x-2)^2+y^2)^{1.5}}. \tag{3.7}$$

$$\text{For } y < 0, \qquad \phi_{yy} = \frac{1}{\sqrt{(x-1)^2+y^2}} - \frac{y^3}{((x-1)^2+y^2)^{1.5}}. \tag{3.8}$$

This shows that $\boldsymbol{\phi_{yy}}$ **has a discontinuity across the ray $x \equiv 0$**.

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[5] Namely: $\phi_{x,x}$, $\phi_{x,y}$, $\phi_{y,x}$, and $\phi_{y,y}$.

Example 2. **Define ϕ for $x > 0$ by**

$$\phi = x \quad \text{for} \;\; y \geq 0, \quad \text{and} \quad \phi = \delta + \sqrt{(x-\delta)^2 + y^2} \qquad \text{for} \;\; y \leq 0, \tag{3.9}$$

where $\delta > 0$ is a constant.

THE END.