

## 18.310 Homework 1

Due September 11th at 6PM

**Instructions:** Remember to submit a separate PDF for each question. Do not forget to include a list of your collaborators or to state that you worked on your own.

1. **Writing Assignment** Following the discussion in recitation, write up a clear and complete proof of the following theorem. Your proof should be easily understandable by another 18.310 student. Your solution *must* be word processed, using either L<sup>A</sup>T<sub>E</sub>X, Word, or some other word processor, and submitted as a pdf file (no Word files allowed). Half of the marks for this question will be allocated to the writing as opposed to simply mathematical correctness. Have a look at the resources linked from the course website for some examples and suggestions. See in particular the notes on “guiding text”.

**Theorem 1.** *A box contains 7 black balls and 4 white balls. Suppose that we repeatedly draw a ball at random from the box, observe its color and then discard it. We do this 4 times. For  $i \in \{1, 2, 3, 4\}$ , let  $X_i$  be the random variable representing the color of the  $i$ th ball drawn from the box. Then,*

$$\mathbb{P}(\{X_1, X_2\} = \{\text{WHITE}, \text{BLACK}\}) = \mathbb{P}(\{X_3, X_4\} = \{\text{WHITE}, \text{BLACK}\}).$$

2. Let  $X$  be a uniformly random subset of  $\{1, 2, \dots, n\}$  (there are  $2^n$  possible subsets, and each is chosen with probability  $1/2^n$ ). Let  $Y$  be another independently chosen random subset.
  - (a) For  $i \in \{1, 2, \dots, n\}$ , let  $A_i$  be the event that  $i \in X$ , and similarly let  $B_i$  be the event that  $i \in Y$ . Argue that the  $2n$  events  $A_1, A_2, \dots, A_n, B_1, B_2, \dots, B_n$  are all independent.

Also determine:

- (b) the probability that  $X \cap Y = \emptyset$ ,
- (c)  $\mathbb{P}(X \cap Y = \emptyset \text{ and } X \text{ has } k \text{ elements})$ , where  $k$  is an integer between 1 and  $n$ .

**Remark:** finding (b) using (c) might not be the simplest solution.

3. A coin is tossed 10 times. This will give a sequence of H’s and T’s, such as HTTHHTHTTT.
  - (a) What is the expected number of times TTH appears consecutively in this sequence.
  - (b) Use the inclusion-exclusion formula to compute the probability that TTH appears (consecutively) at least once in the sequence.
4. Suppose  $N$  people enter an elevator in the basement of a building with  $K$  floors above it. Assume each person gets off at a floor with probability  $1/K$  and independent of any other person. What is the expected number of stops the elevator will make in unloading everyone?

The following exercises should **not** be handed in, but we nevertheless encourage you to do them.

1. (a) We have a box with two black balls and one white ball. We play a game in which we win if we draw the white ball. We draw a ball at random from the box. Before we can see which color it is, the ball is taken away from us and we draw another ball at random from the box. Suppose that we see this second ball and it happens to be black. At this point, we may choose either the first ball we drew or the ball remaining in the box. Which should we choose? **Hint:** Prove that both strategies have the same success probability.  
(b) Consider the same setup, but now the second ball is not drawn at random, but selected to be black (i.e. someone opens the box, finds a black ball and removes it). Do both strategies still have the same success probability? Prove or disprove. **Hint:** What is the relevant subset of the sample space?
2. You roll two dice, each having numbers 1 to 6 on its faces. Consider the following events:
  - (a) A: The number on the first die is even.
  - (b) B: The number on the second die is 1.

Find:

- (a) an event C so that A, B and C are 3 independent events.
- (b) an event D so that A, B and D are 3 events that are pairwise independent but not independent.

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