Fall 2013

18.310 Homework 6

In this writing assignment, you are each assigned one problem below (out of three), and your task is to write a formal solution as if this was an article for an undergraduate mathematics journal. You will need to decide how to structure the article and how to guide your readers while remaining appropriately formal and concise. If you find it appropriate, you are welcome to discuss additional extensions to the question asked (such as generalizations or open questions). The length of the paper is not fixed; you should try to give justice to the problem being solved. Nevertheless, in no case can the paper be longer than 10 pages or shorter than 5 pages.

If you have questions about journal article structure, communicating clearly and formally, etc., ask at office hours or e-mail one of us to arrange a time to meet. You are also encouraged to refer to the writing resources on the course materials page. These resources include an example of an article from an undergraduate mathematics journal. Your article will be reviewed by your peers.

Assignment of problems and schedule.

- Look at the first letter of your lastname in the table below to see which problem you have to write about, and to see the deadline for the first draft.

Course Lastname	Problem	Due date for draft
S, T, W, Y, Z	1	Thu Oct 24th, 6PM
A, B, D, G	2	Thu Oct 24th, 6PM
C, H, J	2	Wed Oct 30th, 6PM
K, L, N, O	3	Wed Oct 30th, 6PM
M, P, R, X	3	Wed Oct 30th, 6PM

  The draft need not be complete and polished, but if content is missing, include a summary of what you plan to include. You will receive feedback from the course staff on the mathematics and the structure, so the more complete the draft, the more helpful the feedback can be.

- **Fri Nov 1st, noon:** (Quiz 2.)
- **Wed Nov 13th, 6PM:** Complete article due, with revisions based on feedback received from the course staff. You will also need to email your paper to two classmates we will assign to you, also by 6PM on Nov 13th.
- **Wed Nov 20th, 6PM:** Critique of articles of two classmates due.
- **Fri Dec 6, noon:** (Quiz 3.)
- **Wed Dec 12, 6PM** : Final version of article due. c

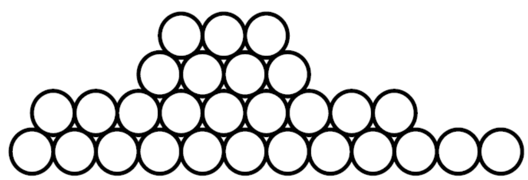
Problems.

1. A *stack of cans* (see Figure 1) refers to an arrangement of n (equal-sized) cans in rows such that (i) the cans in *each* row form a single continuous block, and (ii) any can, except the ones in the bottom row, touches exactly two cans from the row beneath it. If the bottom row contains k cans, this is a (n, k) stack of cans. Figure 1 shows a (28,12) stack of cans.

   Your article should give a formula (as a function of k) for the number s_k of stack of cans with exactly k cans in the bottom row (and an arbitrary total number of cans). Your article should at least contain:

   - A generating function for $(s_k)_{k\geq 0}$,
   - A closed-form formula for s_k for any $k \geq 0$,
   - A relation between s_k and another well-known sequence of numbers.

   If you can find a bijective proof for s_k, you could also provide it. This is however not compulsory and should not replace any of the items above.



2. Recall the inclusion / exclusion formula, let $A_1, A_2, \dots, A_n$ be events, and

$$S_k := \sum_{1\leq i_1<i_2<\cdots<i_k\leq n} \mathbb{P}\left(A_{i_1} \bigwedge A_{i_2} \bigwedge \dots \bigwedge A_{i_k}\right).$$

   The inclusion/exclusion formula gives

$$\mathbb{P}\left(\bigvee_{i=1}^{n} A_i\right) = \sum_{j=1}^{n} (-1)^{j-1}\, S_j.$$

   In this problem we will develop and use an extension of it.

   - Show that for all odd k in $\{1, \dots, n\}$,

$$\mathbb{P}\left(\bigvee_{i=1}^{n} A_i\right) \leq \sum_{j=1}^{k} (-1)^{j-1}\, S_j,$$

     and describe how the union bound is a special case of this.
   - Show that for all even k in $\{1, \dots, n\}$,

$$\mathbb{P}\left(\bigvee_{i=1}^{n} A_i\right) \geq \sum_{j=1}^{k} (-1)^{j-1}\, S_j.$$

- Recall the problem from homework 1 of flipping m fair coins and bounding the probability of a certain string appearing at least once. Use the above inequality to derive tighter bounds for the probability of the sequence $HHH...HT$ containing $l-1$ heads and 1 tail in that order appearing at least once.
  - Show that this probability is at least
  $$\frac{m-l+1}{2^l} - \binom{m-2l+2}{2}\frac{1}{2^{2l}}.$$
  - Show that this probability is at most
  $$\frac{m-l+1}{2^l} - \binom{m-2l+2}{2}\frac{1}{2^{2l}} + \binom{m-3l+3}{3}\frac{1}{2^{3l}}.$$
  - Also provide tighter upper and lower bounds.

  You could also provide additionals results, such as compare when the upper bounds are tighter than the one using the union bound.

3. At some point during the baseball season, the n teams of the American League have already played several games. Suppose team i has won w_i games so far, and $g_{ij} = g_{ji}$ is the number of games that teams i and j have yet to play. No game ends in a tie, so each game yet to be played gives one point to one of the teams and zero point to the other. You would like to decide if your favorite team (Red Sox?), say team n, can still win. In other words, you would like to determine whether there exists an outcome of the games to be played (remember, with no ties) such that team n has at least as many victories as any other team (we allow team n to be tied for first place).

   Your article should contain at least the following:

   - some motivation indicating that the problem is not as trivial as it appears,
   - a construction (with proof and explanations) that this problem can be cast and solved as a maximum flow problem,
   - a theorem (with proof and explanations) that team n can win *if and only if* there is no set $S \subseteq \{1, \cdots, n-1\}$ such that
   $$w_n + \sum_{i=1}^{n-1} g_{in} < \frac{\sum_{i\in S} w_i + \sum_{i<j, i\in S, j\in S} g_{ij}}{|S|}.$$

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