Pak
12/9/05

TPT $\bar{d} = (\ldots d_{-1}\ d_0\ d_1 \ldots)$

$|e(P_{\bar{d}})| = |e(Q_{\bar{d}})|$

$P_{\bar{d}} = C(\lambda) \cap \{\alpha_c = d_c \ \forall c \in \mathbb{Z}\}$ $Q_{\bar{d}} = D(\lambda) \cap \{\beta_c = d_c \ \forall c \in \mathbb{Z}\}$

Cor 1 HLF $|SYT(\lambda)| = \frac{n!}{\prod h_x}$

Cor 2 HCF $\sum_{A \in RPP(\lambda)} t^{|A|} = \prod \frac{1}{1-t^{h_x}}$

Recent example

$d = (1\ 2 \ldots n \ldots 1)$ $\lambda = (n \ldots n)$

TPT $\Rightarrow$ $|S_n| = n! = \sum_{\lambda \vdash n} |SYT(\lambda)|^2$

↳ partitions of n, i.e. diagrams w/ n □'s

Thm $\sum_{\lambda \vdash n} |SYT(\lambda)| = [\#$ involutions in $S_n]$

$= \sum_{k \leq n/2} \binom{n}{2k} (2k-1)!!$

$= 1 \cdot 3 \cdot 5 \cdot 7 \ldots (2k-1)$

Remember the TPT proof? We constructed $\chi_{\lambda}: P_{\bar{d}} \to Q_{\bar{d}}$ recursively. But maybe, since we used χ_A for some $A \in SYT(\lambda)$, χ can vary.

Lemma: $\chi_A = \chi_B^{\lambda}$

Pf: Claim $\chi_A = \chi_{A'}$ if A, A' differ at exactly 2 ~~corners~~ squares $k, k-1$ $1 < k \leq n$

$k = n \Rightarrow$ ✓

Then by induction ... like in homework

Lemma[2]: Every $A, B \in SYT(\lambda)$ are connected by a sequence of exchanges of $k, k+1$

Pf: HW

So overall $\chi_A = \chi_B$ ✓

Example continued

$\lambda = (n \dots n)$ $\qquad \chi_\lambda : A \to B \qquad A \in P_{\vec d},\ B \in Q_{\vec d}$

$\vec d = (d_{1-n} \dots d_{-1}\ d_0 \dots d_{n-1})$

$\vec d^* = (d_{n-1} \dots d_0 \dots d_{1-n})$

$A^T \in P_{\vec d^*}$, $B^T \in Q_{\vec d^*}$

Thm "~~let's call it a theorem,~~ we can call anything a theorem"

$\chi_\lambda : A \to B \qquad \chi_\lambda : A^T \to B^T$

Pf: Let $x \in SYT(\lambda)$ be $\begin{smallmatrix}1 & \dots & n \\ 2n & \dots & 2n\end{smallmatrix}$, $\chi_x : A \to B$

b/c $\chi_\lambda = \chi_x$, and $\Rightarrow \chi_{x^T} : A^T \to B^T$,

and $\chi_\lambda = \chi_{x^T} \Rightarrow \chi_\lambda : A^T \to B^T$.

Suppose $d = 12 \dots n\ n-1 \dots 1$

Then $e(Q_{\vec d})$ = set of permutation matrices. ~~Which~~

Look at $\{B \in e(Q_{\vec d}) \mid B^T = B\}$. That's involutions!

Look at $\{A \in e(P_{\vec d}) \mid A^T = A\}$ remember

$A \in e(P_{\vec d}) \Rightarrow A$ corresponds to a pair $(u, v) \in SYT(\lambda)$

for some $\lambda \vdash n$. A^T corresponds to (v, u), so

$|\{A \in e(P_{\vec d}) \mid A^T = A\}| = \sum_{\lambda \vdash n} |SYT(\lambda)|$

Also $\chi : A \leftrightarrow B \qquad A^T = A \Leftrightarrow (\chi : A^T \leftrightarrow B^T)$ b/c

$B^T = B$. So we've got a bijection between the two sets. ✓

$$\frac{\#\text{involutions in } S_n}{n!} \neq \frac{P(n)}{n!} = Pr(\sigma w = w \sigma) \quad \sigma, w \in S_n$$

($P(n)$ → partitions of n)

Now this old hwk follows immediately! !