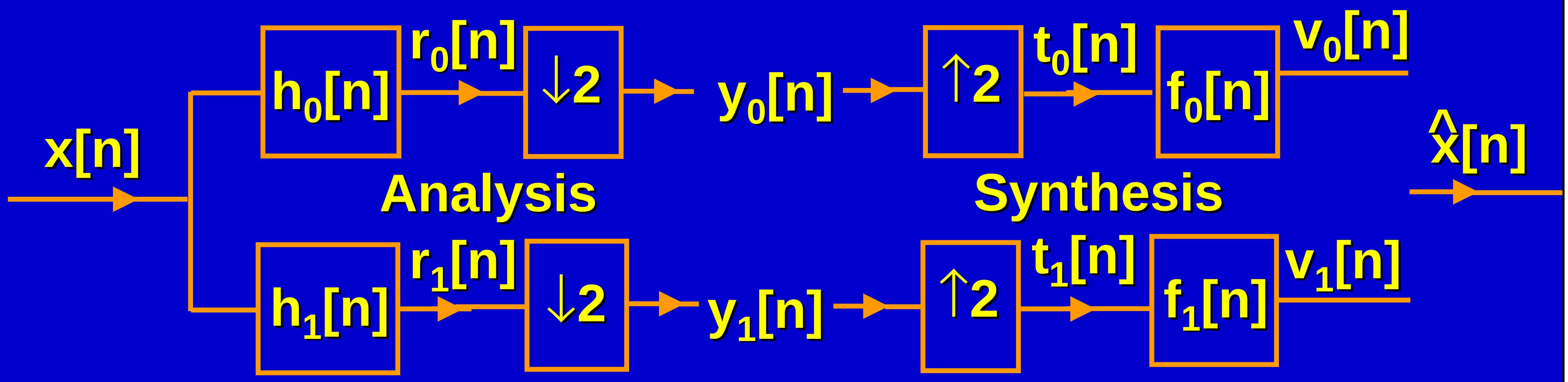
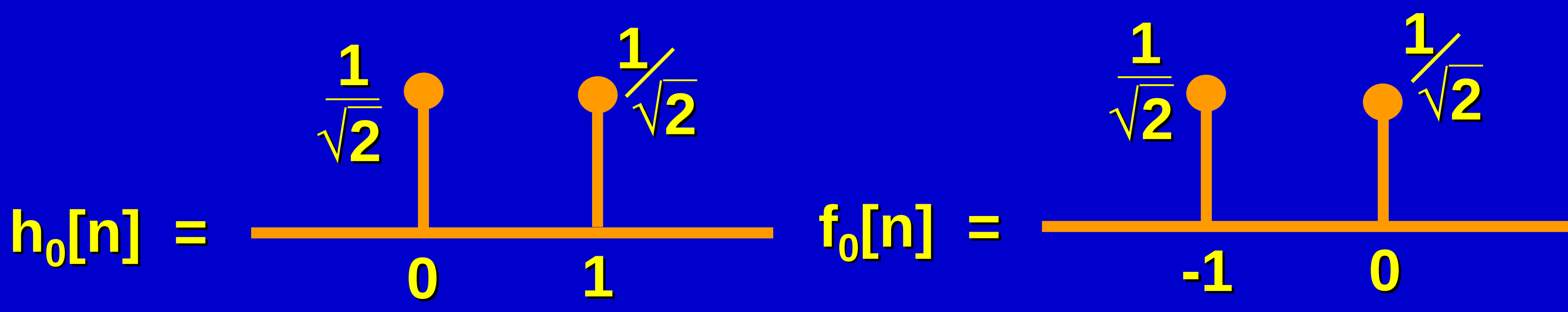
Course 18.327 and 1.130
Wavelets and Filter Banks

Filter Banks: time domain (Haar example) and frequency domain; conditions for alias cancellation and no distortion

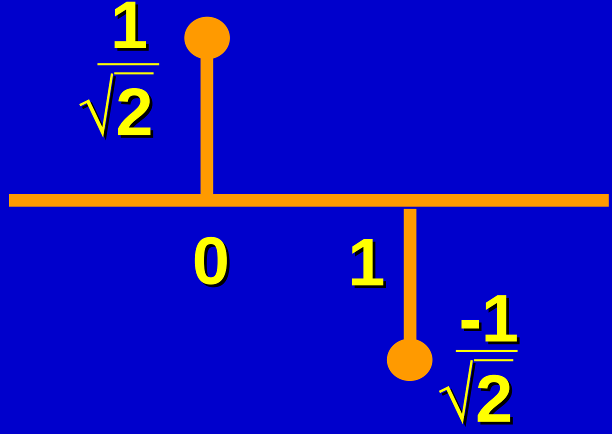
Haar Filter Bank

Simplest (non-trivial) example of a two channel FIR perfect reconstruction filter bank.

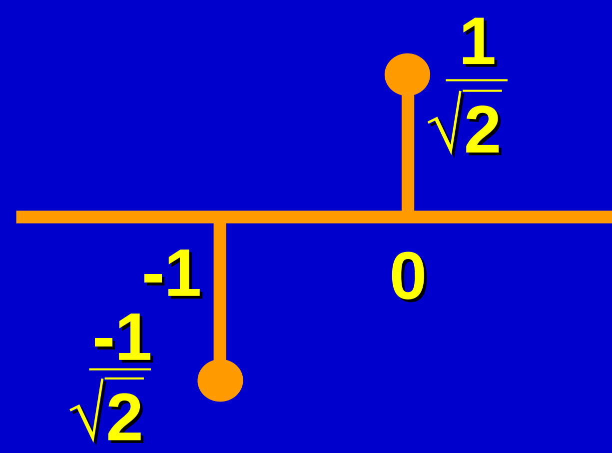




$h_1[n] =$



$f_1[n] =$



Analysis:

$$r_0[n] = \frac{1}{\sqrt{2}}\,(x[n] + x[n-1])$$ **lowpass filter**

$$y_0[n] = r_0[2n]$$ **downsampler**

$$y_0[n] = \frac{1}{\sqrt{2}}\,(x[2n] + x[2n-1])$$ ------------------ j

Similarly

$$y_1[n] = \frac{1}{\sqrt{2}}\,(x[2n] - x[2n-1])$$ ------------------ k

Matrix form

$$
\begin{bmatrix} \vdots \\ y_0[0] \\ y_0[1] \\ \vdots \\ \hline \vdots \\ y_1[0] \\ y_1[1] \\ \vdots \end{bmatrix}
= \frac{1}{\sqrt{2}}
\begin{bmatrix}
 & & & \vdots & & & \\
\cdots & 1 & 1 & & 0 & 0 & \cdots \\
\cdots & 0 & 0 & & 1 & 1 & \cdots \\
\hline
 & & & \vdots & & & \\
\cdots & -1 & 1 & & 0 & 0 & \cdots \\
\cdots & 0 & 0 & & -1 & 1 & \cdots \\
 & & & \vdots & & &
\end{bmatrix}
\begin{bmatrix} \vdots \\ x[-1] \\ x[0] \\ x[1] \\ x[2] \\ \vdots \\ \vdots \end{bmatrix}
$$

$$
\begin{bmatrix} y_o \\ \hline y_1 \end{bmatrix} = \begin{bmatrix} L \\ \hline B \end{bmatrix} x
$$

Synthesis

$$t_0[n] = \begin{cases} y_0[n/2] & n \text{ even} \\ 0 & n \text{ odd} \end{cases} \qquad \text{upsampler}$$

$$v_0[n] = \tfrac{1}{\sqrt{2}} \left(t_0[n+1] + t_0[n] \right) \qquad \text{lowpass filter}$$

$$= \begin{cases} \frac{1}{\sqrt{2}} y_0[n/2] & n \text{ even} \\ \frac{1}{\sqrt{2}} y_0\left[\frac{n+1}{2}\right] & n \text{ odd} \end{cases}$$

Similarly

$$v_1[n] = \begin{cases} \frac{1}{\sqrt{2}}\, y_1[n/2] & \text{n even} \\ \frac{1}{\sqrt{2}}\, y_1[\frac{n+1}{2}] & \text{n odd} \end{cases}$$

So, the reconstructed signal is

$$\hat{x}[n] = v_0[n] + v_1[n]$$

$$= \begin{cases} 1/\sqrt{2}\, (y_0[n/2] + y_1[n/2]) & \text{n even} \\ 1/\sqrt{2}\, (y_0[\frac{n+1}{2}] - y_1[\frac{n+1}{2}]) & \text{n odd} \end{cases}$$

i.e.

$$\hat{x}[2n-1] = \frac{1}{\sqrt{2}} (y_0[n] - y_1[n]) = x[2n-1]$$

from j and k

$$\hat{x}[2n] = \frac{1}{\sqrt{2}} (y_0[n] + y_1[n]) = x[2n]$$

So $\hat{x}[n] = x[n] \Rightarrow$ Perfect reconstruction!

In general, we will make all filters causal, so we will have

$$\hat{x}[n] = x[n - n_0] \Rightarrow \text{PR with delay}$$

Matrix form

$$\begin{bmatrix} \vdots \\ \hat{x}[-1] \\ \hat{x}[0] \\ \\ \hat{x}[1] \\ \hat{x}[2] \\ \vdots \end{bmatrix} = \frac{1}{\sqrt{2}} \left[\begin{array}{cc|cc} \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 1 & 1 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 1 & 0 & -1 \\ 0 & 1 & -1 & 1 \\ \vdots & \vdots & \vdots & \vdots \end{array}\right] \begin{bmatrix} \vdots \\ y_0[0] \\ y_0[1] \\ \vdots \\ \hline \vdots \\ y_1[0] \\ y_1[1] \\ \vdots \end{bmatrix}$$

$$\hat{x} = \left[\begin{array}{c|c} L^T & B^T \end{array}\right] \begin{bmatrix} y_0 \\ \hline y_1 \end{bmatrix}$$

----------------m

Perfect reconstruction means that the synthesis bank is the inverse of the analysis bank.

$$\hat{x} = x \Rightarrow \underbrace{\begin{bmatrix} L^T & \Big| & B^T \end{bmatrix}}_{W^{-1}} \underbrace{\begin{bmatrix} L \\ \hline B \end{bmatrix}}_{W} = I$$

1 2 3 1 2 3

(Wavelet transform matrix)

In the Haar example, we have the special case

$$W^{-1} = W^T \rightarrow \text{orthogonal matrix}$$

So we have an orthogonal filter bank, where

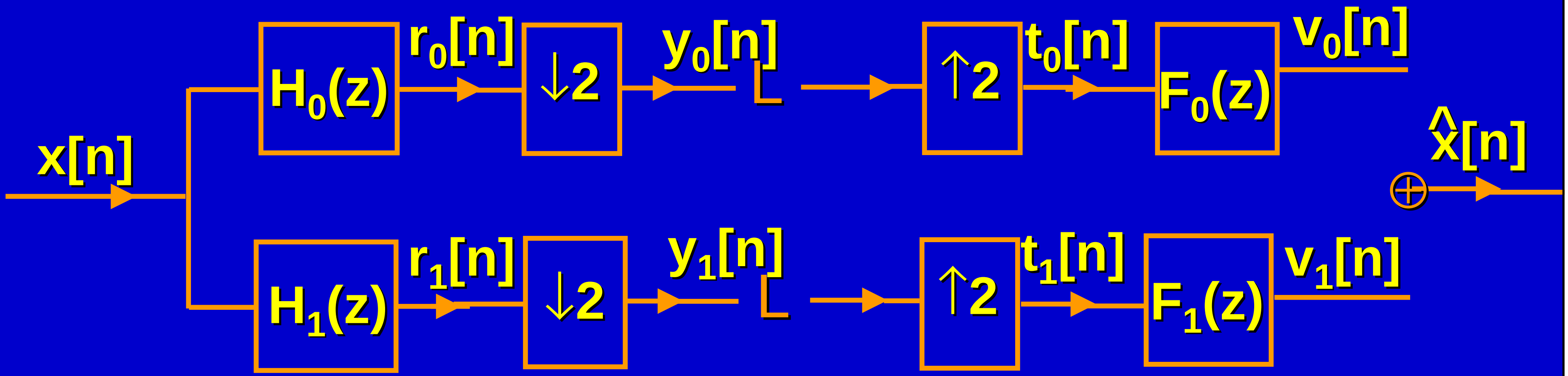
Synthesis bank = Transpose of Analysis bank

$$f_0[n] = h_0[-n]$$

$$f_1[n] = h_1[-n]$$

Perfect Reconstruction Filter Banks

General two-channel filter bank



z-transform definition:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

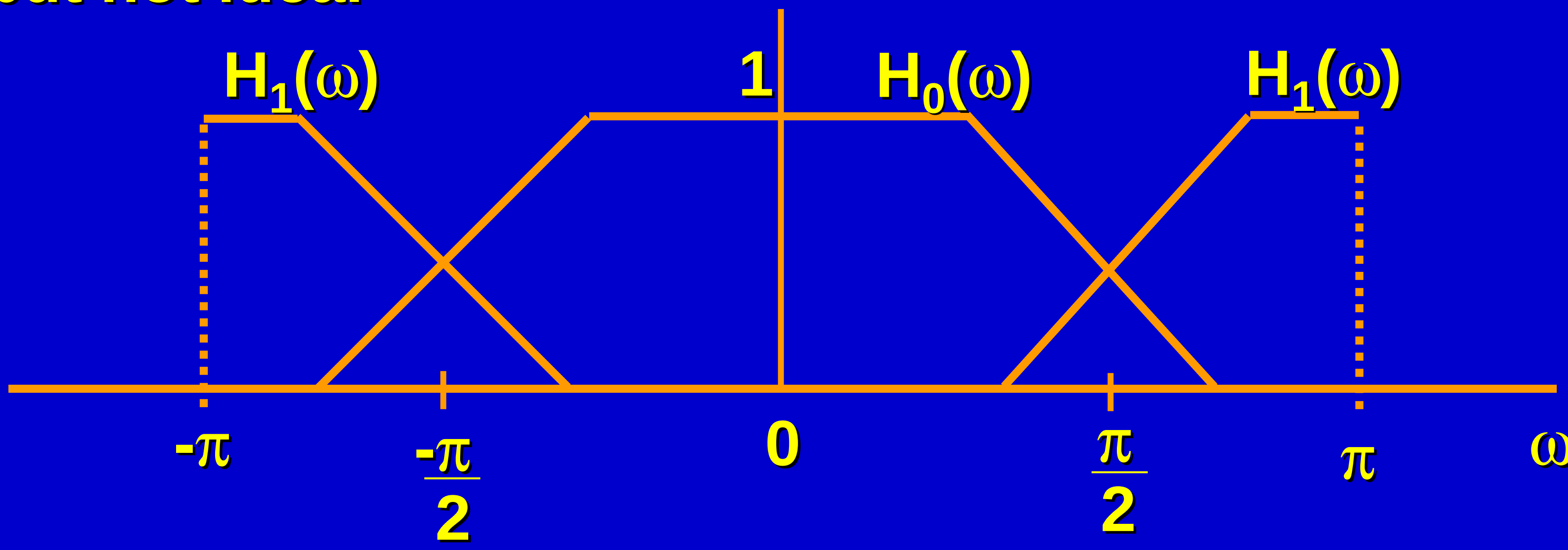
Put $z = e^{i\omega}$ to get DTFT

Perfect reconstruction requirement:

$$\hat{x}[n] = x[n - l] \quad (l \text{ time delays})$$

$$\hat{X}(z) = z^{-l} X(z)$$

$H_0(z)$ and $H_1(z)$ are normally lowpass and highpass, but not ideal



$\Rightarrow$ Downsampling operation in each channel can produce aliasing

Let's see why:

Lowpass channel has

$$Y_0(z) = \tfrac{1}{2}\{R_0(z^{1/2}) + R_0(-z^{1/2})\} \qquad \text{(downsampling)}$$

$$= \tfrac{1}{2}\{H_0(z^{1/2})\, X(z^{1/2}) + H_0(-z^{1/2})X(-z^{1/2})\}$$

In frequency domain:

$$X(z) \rightarrow X(\omega) \qquad \text{or } X(e^{i\omega})$$

$$X(-z) \rightarrow X(\omega + \pi)$$

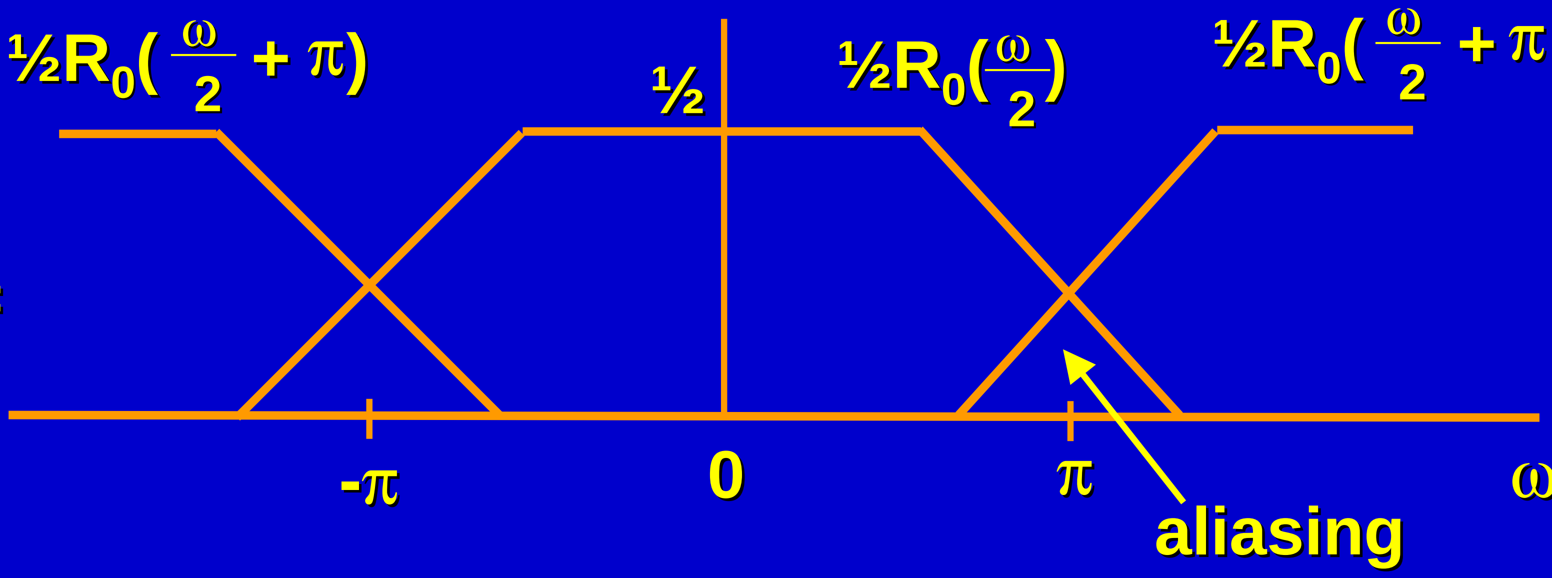
$$X(z^{1/2}) \rightarrow X(\tfrac{\omega}{2})$$

$$Y_0(\omega) = \tfrac{1}{2}\{H_0(\tfrac{\omega}{2})\, X(\tfrac{\omega}{2}) + H_0(\tfrac{\omega}{2} + \pi)X(\tfrac{\omega}{2} + \pi)\}$$

Suppose $X(\omega) = 1$ (input has all frequencies)

Then $R_0(\omega) = H_0(\omega)$, so that after downsampling we have

$Y_0(\omega) =$



Goal is to design $F_0(z)$ and $F_1(z)$ so that the overall system is just a simple delay - with no aliasing term:

$$V_0(z) + V_1(z) = z^{-l}X(z)$$

$$V_0(z) \;=\; F_0(z)\,T_0(z)$$

$$=\; F_0(z)\,Y_0(z^2) \qquad \text{(upsampling)}$$

$$=\; \tfrac{1}{2}F_0(z)\{\,H_0(z)\,X(z) \;+\; H_0(-z)\,X(-z)\}$$

$$V_1(z) \;=\; \tfrac{1}{2}F_1(z)\{\,H_1(z)\,X(z) \;+\; H_1(-z)\,X(-z)\}$$

So we want

$$\tfrac{1}{2}\,\{F_0(z)\,H_0(z) + F_1(z)\,H_1(z)\,\}\,X(z)$$

$$+ \qquad\qquad = \; z^{-l}\,X(z)$$

$$\tfrac{1}{2}\,\{F_0(z)\,H_0(-z) + F_1(z)\,H_1(-z)\,\}\,X(-z)$$

Compare terms in X(z) and X(-z):

1) Condition for no distortion (terms in X (z) amount to a delay)

$$F_0(z)\,H_0(z) \;+\; F_1(z)\,H_1(z) \;=\; 2z^{-l} \qquad \text{--------------- j}$$

2) Condition for alias cancellation (no term in X(-z))

$$F_0(z)\,H_0(-z) \;+\; F_1(z)\,H_1(-z) \;=\; 0 \qquad \text{--------------- k}$$

To satisfy alias cancellation condition, choose

$$F_0(z) \;=\; H_1(-z)$$
$$F_1(z) \;=\; -H_0(-z) \qquad \text{------------------------ l}$$

What happens in the time domain?

$$F_0(z) = H_1(-z) \qquad F_0(\omega) = H_1(\omega + \pi)$$

$$= \sum_n h_1[n] \, (-z)^{-n}$$

$$= \sum_n (-1)^n \, h_1[n] \, z^{-n}$$

So the filter coefficients are

$$f_0[n] = (-1)^n \quad h_1[n]$$

$$f_1[n] = (-1)^{n+1} \, h_0[n]$$

alternating signs rule

Example

$$h_0[n] = \{ a_0, a_1, a_2 \} \qquad f_0[n] = \{ b_0, -b_1, b_2 \}$$

$$h_1[n] = \{ b_0, b_1, b_2 \} \qquad f_1[n] = \{ -a_0, a_1, -a_2 \}$$

Product Filter

Define

$$P_0(z) = F_0(z)\,H_0(z) \quad \text{-------------------------------} \; m$$

Substitute $F_1(z) = -H_0(-z)$, $H_1(z) = F_0(-z)$

in the zero distortion condition (Equation j)

$$F_0(z)\,H_0(z) - F_0(-z)\,H_0(-z) = 2z^{-l}$$

i.e.

$$P_0(z) - P_0(-z) = 2z^{-l} \quad \text{-----------------------------} \; n$$

Note: l must be odd since LHS is an odd function.

Normalized Product Filter

Define

$$P(z) = z^{l} P_0(z)$$ ------------------------------ o

$$P(-z) = -z^{l} P_0(-z)$$ since l is odd

So we can rewrite Equation n as

$$z^{-l} P(z) + z^{-l} P(-z) = 2z^{-l}$$

i.e. $$P(z) + P(-z) = 2$$ ---------------------------- p

This is the condition on the normalized product filter for Perfect Reconstruction.

Design Process

1. Design P(z) to satisfy Equation p. This gives $P_0(z)$. Note: P(z) is designed to be lowpass.
2. Factor $P_0(z)$ into $F_0(z)\,H_0(z)$. Use Equations l to find $H_1(z)$ and $F_1(z)$.

Note: Equation p requires all even powers of z (except z^0) to be zero:

$$\sum_n p[n]z^{-n} + \sum_n p[n](-z)^{-n} = 2$$

$$\Rightarrow \; p[n] = \begin{cases} 1 & ; \quad n = 0 \\ 0 & ; \quad \text{all even } n \; (n \neq 0) \end{cases}$$

For odd n, p[n] and –p[n] cancel.

The odd coefficients, p[n], are free to be designed according to additional criteria.

Example: Haar filter bank

$$H_0(z) = \frac{1}{\sqrt{2}}(1 + z^{-1}) \qquad H_1(z) = \frac{1}{\sqrt{2}}(1 - z^{-1})$$

$$F_0(z) = H_1(-z) = \frac{1}{\sqrt{2}}(1 + z^{-1})$$

$$F_1(z) = -H_0(-z) = \frac{-1}{\sqrt{2}}(1 - z^{-1})$$

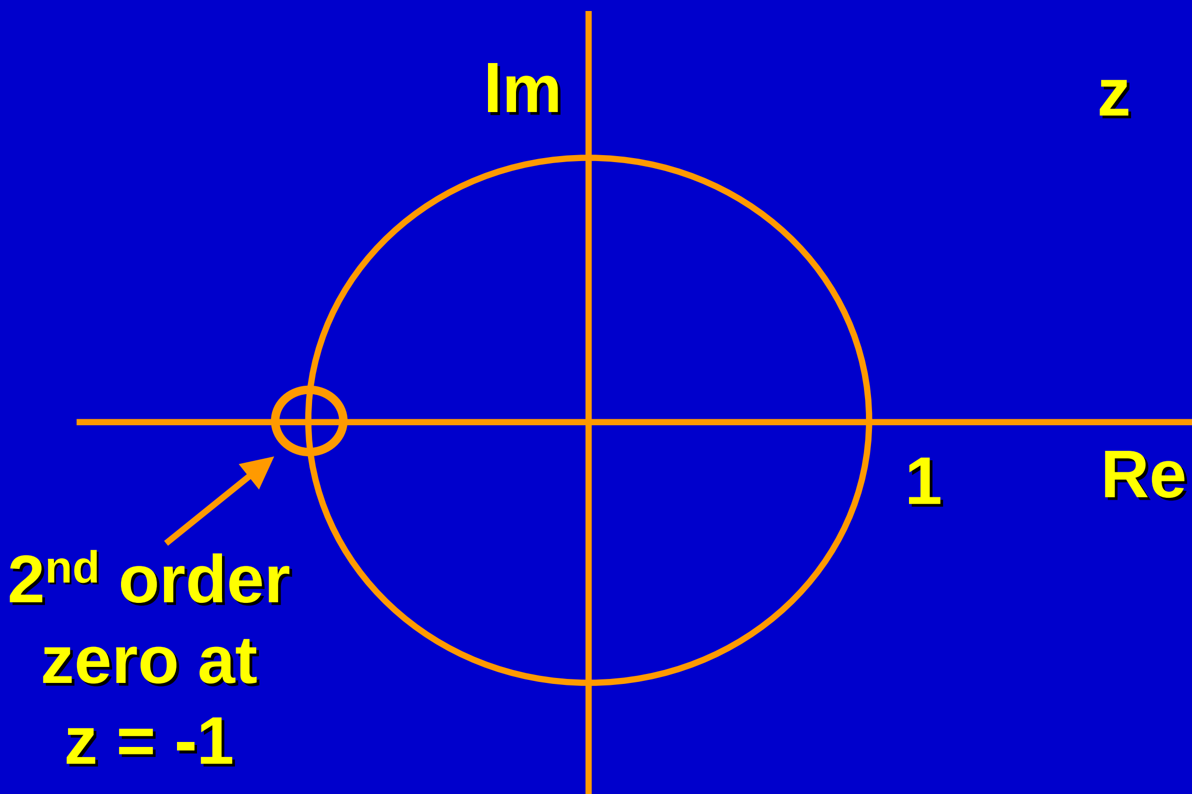
$$P_0(z) = F_0(z)\,H_0(z) = \frac{1}{2}(1 + z^{-1})^2$$

So the Perfect Reconstruction requirement is

$$P_0(z) - P_0(-z) = \frac{1}{2}(1 + 2z^{-1} + z^{-2}) - \frac{1}{2}(1 - 2z^{-1} + z^{-2})$$

$$= 2z^{-1} \quad \Rightarrow \quad l = 1$$

$$P(z) = z^{l} P_0(z) = \frac{1}{2}(1 + z)(1 + z^{-1})$$



Zeros of P(z):

$1 + z = 0$

$1 + z^{-1} = 0$