SECOND ORDER FREENESS AND FLUCTUATIONS OF RANDOM MATRICES: II. UNITARY RANDOM MATRICES

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Abstract. We extend the relation between random matrices and free probability theory from the level of expectations to the level of fluctuations. We show how the concept of "second order freeness", which was introduced in Part I, allows one to understand global fluctuations of Haar distributed unitary random matrices. In particular, independence between the unitary ensemble and another ensemble goes in the large N limit over into asymptotic second order freeness. As a corollary, this also yields a generalization of a theorem of Diaconis and Shahshahani to the case of several independent unitary matrices.

1. Introduction

In Part I of this series [MSp] we introduced the concept of second order freeness as the mathematical concept for dealing with the large N limit of fluctuations of $N \times N$-random matrices. Whereas Voiculescu's freeness (of first order) provides the crucial notion behind the leading order of expectations of traces, our second order freeness is intended to describe in a similar way the structure of leading orders of global fluctuations, i.e., of variances of traces. In Part I we showed how fluctuations of Gaussian and Wishart random matrices can be understood from this perspective. Here we want to aim at a corresponding treatment for fluctuations of unitary random matrices. Global fluctuations of unitary random matrices have received much attention in the last decade, see, e.g, the survey article of Diaconis [D].

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Our main concern will be to understand the relation between unitary random matrices and some other ensemble of random matrices which is independent from the unitary ensemble. This includes in particular the case that the second ensemble consists of constant (i.e., non-random) matrices. A basic result of Voiculescu tells us that on the level of expectations, independence between the ensembles goes over into asymptotic freeness. We will show that this result remains true on the level of fluctuations: Independence between the ensembles implies that we have asymptotic second order freeness between their fluctuations.

As a byproduct of these results we also get a generalization to the case of several independent unitary random matrices of a classical result of Diaconis and Shahshahani [DS]. Their one-dimensional case states that, for a unitary random matrix U, the family of traces $\mathrm{Tr}(U^n)$ converge towards a Gaussian family where the covariance between $\mathrm{Tr}(U^m)$ and $\mathrm{Tr}(U^{*n})$ is given by $n \cdot \delta_{mn}$. In the case of several independent unitary random matrices, one has to consider traces in reduced words of these random matrices, and again these converge to a Gaussian family, where the covariance between two such reduced words is now given by the number of cyclic rotations which match one word with the other. This result was also independently derived by Rădulescu [R] in the course of his investigations around Connes's embedding problem.

The paper is organized as follows. In Section 2, we recall all the necessary definitions and results around permutations, unitary random matrices, and second order freeness. We will recall all the relevant notions from Part I, so that our presentation will be self-contained. However, for getting more background information on the concept of second order freeness one should consult [MSp]. In Section 3, we derive our main result about the asymptotic second order freeness between unitary random matrices and another independent random matrix ensemble. This yields as corollary that independent unitary random matrices are asymptotically free of second order, implying the above mentioned generalization of the result of Diaconis and Shahshahani [DS].

2. Preliminaries

2.1. **Some general notation.** For natural numbers $m, n \in \mathbb{N}$ with $m < n$, we denote by $[m, n]$ the interval of natural numbers between m and n, i.e.,

$$[m, n] := \{m, m + 1, m + 2, \ldots, n - 1, n\}.$$

For a matrix $A = (a_{ij})_{i,j=1}^N$, we denote by Tr the unnormalized and by tr the normalized trace,

$$\mathrm{Tr}(A) := \sum_{i=1}^{N} a_{ii}, \qquad \mathrm{tr}(A) := \frac{1}{N}\mathrm{Tr}(A).$$

If we are considering classical random variables on some probability space, then we denote by E the expectation with respect to the corresponding probability measure and by k_r the corresponding classical cumulants (as multi-linear functionals in r arguments); in particular,

$$\mathrm{k}_1\{a\} = \mathrm{E}\{a\} \qquad \text{and} \qquad \mathrm{k}_2\{a_1, a_2\} = \mathrm{E}\{a_1 a_2\} - \mathrm{E}\{a_1\}\mathrm{E}\{a_2\}.$$

2.2. **Permutations.** We will denote the set of permutations on n elements by S_n. We will quite often use the cycle notation for such permutations, i.e., $\pi = (i_1, i_2, \dots, i_r)$ is a cycle which sends i_k to i_{k+1} $(k = 1, \dots, r)$, where $i_{r+1} = i_1$.

2.2.1. *Length function.* For a partition $\pi \in S_n$ we denote by $\#\pi$ the number of cycles of π and by $|\pi|$ the minimal number of transpositions needed to write π as a product of transpositions. Note that one has

$$|\pi| + \#\pi = n \qquad \text{for all } \pi \in S_n.$$

2.2.2. *Non-crossing permutations.* Let us denote by $\gamma_n \in S_n$ the cycle

$$\gamma_n = (1, 2, \dots, n).$$

For all $\pi \in S_n$ one has that

$$|\pi| + |\gamma_n \pi^{-1}| \le n - 1.$$

If we have equality then we call π *non-crossing*. Note that this is equivalent to

$$\#\pi + \#(\gamma_n \pi^{-1}) = n + 1.$$

If π is non-crossing, then so are $\gamma_n \pi^{-1}$ and $\pi^{-1}\gamma_n$; the latter is called the *(Kreweras) complement* of π.

We will denote the set of non-crossing permutations in S_n by $NC(n)$. Note that such a non-crossing permutation can be identified with a non-crossing partition, by forgetting the order on the cycles. There is exactly one cyclic order on the blocks of a non-crossing partition which makes it into a non-crossing permutation.

2.2.3. *Annular non-crossing permutations.* Fix $m, n \in \mathbb{N}$ and denote by $\gamma_{m,n}$ the product of the two cycles

$$\gamma_{m,n} = (1, 2, \dots, m)(m+1, m+2, \dots, m+n).$$

More generally, we shall denote by $\gamma_{m_1,\dots,m_k}$ the product of the corresponding k cycles.

We call a $\pi \in S_{m+n}$ *connected* if the pair π and $\gamma_{m,n}$ generates a transitive subgroup in S_{m+n}. A connected permutation $\pi \in S_{m+n}$ always satisfies

$$|\pi| + |\gamma_{m,n}\pi^{-1}| \leq m + n. \tag{1}$$

If π is connected and if we have equality in that equation then we call π *annular non-crossing*. Note that with π also $\gamma_{m,n}\pi^{-1}$ is annular non-crossing. Again, we call the latter the *complement* of π. Of course, all the above notations depend on the pair (m, n); if we want to emphasize this dependency we will also speak about (m, n)-connected permutations and (m, n)-annular non-crossing permutations.

We will denote the set of (m, n)-annular non-crossing permutations by $S_{NC}(m, n)$. Again one can go over to annular non-crossing partitions by forgetting the cyclic orders on cycles; however, in the annular case, the relation between non-crossing permutation and non-crossing partition is not one-to-one. Since we will not use the language of annular partitions in the present paper, this is of no relevance here.

Annular non-crossing permutations and partitions were introduced in [MN]; there, many different characterizations—in particular, the one (1) above in terms of the length function—were given.

2.2.4. *Other notations.* We say that $A = \{A_1, \dots, A_k\}$ is a partition of a set $[1, n]$ if sets $A_i = \{A_{i,1}, \dots, A_{i,l(i)}\}$ are disjoint and non–empty and their union is equal to $[1, n]$. We call $A_1, \dots, A_k$ the blocks of partition A. For a permutation $\pi \in S_n$ we say that a partition A is π-invariant if π preserves each block A_i.

If $A = \{A_1, \dots, A_k\}$ and $B = \{B_1, \dots, B_l\}$ are partitions of the same set, we say that $A \leq B$ if for every block A_i there exists some block B_j such that $A_i \subseteq B_j$. For a pair of partitions A, B we denote by $A \vee B$ the smallest partition C such that $A \leq C$ and $B \leq C$. We denote by $1_{[1,n]} = \big\{[1, n]\big\}$ the biggest partition of the set $[1, n]$.

If, for $1 \leq i \leq k$, π_i is a permutation of the set A_i we denote by $\pi_1 \times \cdots \times \pi_k \in S_n$ the concatenation of these permutations. We say that $\pi = \pi_1 \times \cdots \times \pi_k$ is a cycle decomposition if additionally every factor π_i is a cycle.

2.3. **Haar distributed unitary random matrices and the Weingarten function.** In the following we will be interested in the asymptotics of special matrix integrals over the group $\mathcal{U}(N)$ of unitary $N\times N$-matrices. We always equip the compact group $\mathcal{U}(N)$ with its Haar probability measure and address its elements then as *Haar distributed unitary random matrices*. Thus the expectation E over this ensemble is given by integrating with respect to the Haar measure.

The expectation of products of entries of Haar distributed unitary random matrices can be described in terms of a special function on the permutation group. Since such considerations go back to Weingarten [W], Collins [C] calls this function the Weingarten function and denotes it by Wg. We will follow his notation. In the following we just recall the relevant information about this Weingarten function, for more details we refer to [C, CŚ].

We use the following definition of the Weingarten function. For $\pi\in S_n$ and $N\geq n$ we put

$$\mathrm{Wg}(N,\pi)=\mathrm{E}[U_{11}\cdots U_{nn}\overline{U}_{1\pi(1)}\cdots\overline{U}_{n\pi(n)}],$$

where $U=(U_{ij})_{i,j=1}^N$ is an $N\times N$ Haar distributed unitary random matrix. Sometimes we will suppress the dependency on N and just write $\mathrm{Wg}(\pi)$. This $\mathrm{Wg}(N,\pi)$ depends on π only through its conjugacy class. General matrix integrals over the unitary groups can be calculated as follows:

$$\text{(2)}\quad \mathrm{E}[U_{i_1'j_1'}\cdots U_{i_n'j_n'}\overline{U}_{i_1j_1}\cdots\overline{U}_{i_nj_n}] = \sum_{\alpha,\beta\in S_n}\delta_{i_1i'_{\alpha(1)}}\cdots\delta_{i_ni'_{\alpha(n)}}\delta_{j_1j'_{\beta(1)}}\cdots\delta_{j_nj'_{\beta(n)}}\mathrm{Wg}(\beta\alpha^{-1}).$$

The Weingarten function is a quite complicated object, and its full understanding is at the basis of questions around Itzykson-Zuber integrals. For our purposes, only the behaviour of leading orders in N of $\mathrm{Wg}(N,\pi)$ is important. One knows (see, e.g., [C, CŚ]) that the leading order in $1/N$ is given by $|\pi|+n$ and increases in steps of 2.

Let us use the following notation for the first two orders ($\pi\in S(n)$):

$$\mathrm{Wg}(N,\pi)=\mu(\pi)N^{-(|\pi|+n)}+\phi(\pi)N^{-(|\pi|+n+2)}+O\big(N^{-(|\pi|+n+4)}\big).$$

One knows that μ is multiplicative with respect to the cycle decomposition, i.e.,

$$\mu(\pi_1\times\pi_2)=\mu(\pi_1)\cdot\mu(\pi_2).$$

The important part of the second order information is contained in the leading order of $\mathrm{Wg}(\pi_1\times\pi_2)-\mathrm{Wg}(\pi_1)\mathrm{Wg}(\pi_2)$, which is given by

$$\mu_2(\pi_1,\pi_2):=\mu(\pi_1\times\pi_2)-\mu(\pi_1)\phi(\pi_2)-\phi(\pi_1)\mu(\pi_2).$$

Note that we have

$$\mu_2(\pi_1, \pi_2) = \mu_2(\pi_2, \pi_1).$$

Collins [C] has general counting formulas for the calculation of μ and μ_2 (and also higher order analogues); however, a conceptual explanation of μ_2 seems still to be missing. μ is the Moebius function of the lattice of non-crossing partitions (thus determined by Catalan numbers), and this fact is quite well understood by the relation between μ and asymptotic freeness of unitary random matrices. In a similar way, one should get a conceptual understanding of μ_2 by the relation with second order freeness. In the present paper we will not pursue further this direction, but we will come back to it in forthcoming investigations. Here we will not rely on the concrete values of μ or μ_2, but will only use their above mentioned basic properties.

2.4. **Second order freeness.** In [MSp], we introduced the concept of second order freeness which is intended to capture the structure of the fluctuation functionals for random matrices arising in the limit $N \to \infty$, in the same way as the usual freeness captures the structure of the expectation of the trace in the limit. We recall the relevant notations and definitions.

Definition 2.1. A *second order non-commutative probability space* $(\mathcal{A}, \varphi_1, \varphi_2)$ consists of a unital algebra $\mathcal{A}$, a tracial linear functional

$$\varphi_1 : \mathcal{A} \to \mathbb{C} \qquad \text{with} \qquad \varphi(1) = 1$$

and a bilinear functional

$$\varphi_2 : \mathcal{A} \times \mathcal{A} \to \mathbb{C},$$

which is tracial in both arguments and which satisfies

$$\varphi_2(a, 1) = 0 = \varphi_2(1, b) \qquad \text{for all } a, b \in \mathcal{A}.$$

Notation 2.2. Let unital subalgebras $\mathcal{A}_1, \dots, \mathcal{A}_r \subset \mathcal{A}$ be given.
1) We say that a tuple $(a_1, \dots, a_n)$ $(n \geq 1)$ of elements from $\mathcal{A}$ is *cyclically alternating* if, for each k, we have an $i(k) \in \{1, \dots, r\}$ such that $a_k \in \mathcal{A}_{i(k)}$ and, if $n \geq 2$, we have $i(k) \neq i(k+1)$ for all $k = 1, \dots, n$. We count indices in a cyclic way modulo n, i.e., for $k = n$ the above means $i(n) \neq i(1)$. Note that for $n = 1$, we do not impose any condition on neighbours.
2) We say that a tuple $(a_1, \dots, a_n)$ of elements from $\mathcal{A}$ is *centered* if we have

$$\varphi_1(a_k) = 0 \qquad \text{for all } k = 1, \dots, n.$$

Definition 2.3. Let $(\mathcal{A}, \varphi_1, \varphi_2)$ be a second order non-commutative probability space. We say that unital subalgebras $\mathcal{A}_1, \dots, \mathcal{A}_r \subset \mathcal{A}$ are *free with respect to* (φ_1, φ_2) or *free of second order*, if they are free (in the usual sense [VDN]) with respect to φ_1 and if the following condition for φ_2 is satisfied: Whenever we have, for $n, m \geq 1$, tuples $(a_1, \dots, a_n)$ and $(b_m, \dots, b_1)$ from $\mathcal{A}$ such that both are centered and cyclically alternating then we have

(1) If $n \neq m$, then
$$\varphi_2(a_1 \cdots a_n, b_m \cdots b_1) = 0.$$
(2) If $n = m = 1$ and $a \in \mathcal{A}_i$, $b \in \mathcal{A}_j$, with $i \neq j$, then
$$\varphi_2(a, b) = 0.$$
(3) If $n = m \geq 2$, then
$$\varphi_2(a_1 \cdots a_n, b_n \cdots b_1) = \sum_{k=0}^{n-1} \varphi_1(a_1 b_{1+k}) \cdot \varphi_1(a_2 b_{2+k}) \cdots \varphi(a_n b_{n+k}).$$

For a visualization of this formula, one should think of two concentric circles with the a's on one of them and the b's on the other. However, whereas on one circle we have a clockwise orientation of the points, on the other circle the orientation is counter-clockwise. Thus, in order to match up these points modulo a rotation of the circles, we have to pair the indices as in the sum above.

Recall that in the combinatorial description of freeness [NSp], the extension of φ_1 to a multiplicative function on non-crossing partitions plays a fundamental role. In the same way, second order freeness will rely on a suitable extension of φ_2.

Notation 2.4. Let $(\mathcal{A}, \varphi_1, \varphi_2)$ be a second order non-commutative probability space. Then we extend the definition of φ_1 and φ_2 as follows:
$$\varphi_1 : \bigcup_{n=1}^{\infty} \left(S_n \times \mathcal{A}^n\right) \to \mathbb{C}$$
$$(\pi, a_1, \dots, a_n) \mapsto \varphi_1(\pi)[a_1, \dots, a_n]$$
is, for a cycle $\pi = (i_1, i_2, \dots, i_r)$, given by
$$\varphi_1(\pi)[a_1, \dots, a_n] := \varphi_1(a_{i_1} a_{i_2} a_{i_3} \cdots a_{i_r})$$
and extended to general $\pi \in S_n$ by multiplicativity
$$\varphi_1(\pi_1 \times \pi_2)[a_1, \dots, a_n] = \varphi_1(\pi_1)[a_1, \dots, a_n] \cdot \varphi_1(\pi_2)[a_1, \dots, a_n].$$

In a similar way,

$$\varphi_2 : \bigcup_{m,n=1}^{\infty} \big(S_m \times S_n \times \mathcal{A}^m \times \mathcal{A}^n\big) \to \mathbb{C}$$
$$(\pi_1, \pi_2, a_1, \dots, a_m, b_1, \dots, b_m) \mapsto \varphi_2(\pi_1, \pi_2)[a_1, \dots, a_m; b_1, \dots, b_n]$$

is defined, for two cycles $\pi_1 = (i_1, i_2, \dots, i_p)$ and $\pi_2 = (j_1, j_2, \dots, j_r)$, by

$$\varphi_2(\pi_1, \pi_2)[a_1, \dots, a_m; b_1, \dots, b_n] := \varphi_2(a_{i_1} a_{i_2} \cdots a_{i_p}, b_{j_1} b_{j_2} \cdots b_{j_r})$$

and extended to the general situation by a 'cocycle property'

$$\begin{aligned} &\varphi_2(\pi_1 \times \pi_2, \pi_3)[a_1, \dots, a_m; b_1, \dots, b_n] \\ &= \varphi_2(\pi_1, \pi_3)[a_1, \dots, a_m; b_1, \dots, b_n] \cdot \varphi_1(\pi_2)[a_1, \dots, a_m, b_1, \dots, b_n] \\ &\quad + \varphi_2(\pi_2, \pi_3)[a_1, \dots, a_m; b_1, \dots, b_n] \cdot \varphi_1(\pi_1)[a_1, \dots, a_m, b_1, \dots, b_n]. \end{aligned} \tag{3}$$

and

$$\begin{aligned} &\varphi_2(\pi_1, \pi_2 \times \pi_3)[a_1, \dots, a_m; b_1, \dots, b_n] \\ &= \varphi_2(\pi_1, \pi_2)[a_1, \dots, a_m; b_1, \dots, b_n] \cdot \varphi_1(\pi_3)[a_1, \dots, a_m, b_1, \dots, b_n] \\ &\quad + \varphi_2(\pi_1, \pi_3)[a_1, \dots, a_m; b_1, \dots, b_n] \cdot \varphi_1(\pi_2)[a_1, \dots, a_m, b_1, \dots, b_n]. \end{aligned} \tag{4}$$

3. Asymptotic second order freeness for unitary random matrices

Notation 3.1. Suppose $\epsilon : [2l] \to \{-1, 1\}$ is such that $\sum_{i=1}^{2l} \epsilon_i = 0$. We write $\epsilon^{-1}(1) = \{p_1, p_2, \dots, p_l\}$ and $\epsilon^{-1}(-1) = \{q_1, q_2, \dots, q_l\}$, with $p_1 < p_2 < \cdots < p_l$ and $q_1 < q_2 < \cdots < q_l$. Let $S_{2l}^{(\epsilon)}$ be the permutations π in S_{2l} such that π takes $\{p_1, \dots, p_l\}$ onto $\{q_1, \dots, q_l\}$ and vice versa. Given a π in $S_{2l}^{(\epsilon)}$ we may extract a pair of permutations α_π and β_π in S_l from the equations

$$\pi(p_{\alpha_\pi(k)}) = q_k \text{ and } \pi(q_k) = p_{\beta_\pi(k)}$$

and conversely: $(\alpha, \beta) \mapsto \pi_{\alpha,\beta}$. Thus we have a bijection of sets between $S_{2l}^{(\epsilon)}$ and $S_l \times S_l$.

Given $\pi \in S_{2l}^{(\epsilon)}$ we let $\tilde{\pi} \in S_l$ be defined by

$$\pi^2(p_k) = p_{\tilde{\pi}(k)}$$

Note that $\tilde{\pi}_{\alpha,\beta} = \beta\alpha^{-1}$.

Note that we have

$$\#\pi = \#\tilde{\pi},$$

and thus

$$|\pi| = |\tilde{\pi}| + l.$$

Lemma 3.2. *Fix $l \in \mathbb{N}$ and $\gamma \in S_{2l}$. Let, for $N \in \mathbb{N}$, U be a Haar distributed unitary $N \times N$ random matrix. Let $\epsilon : [2l] \to \{-1, 1\}$ such that $\sum_{i=1}^{2l} \epsilon_i = 0$. Then we have for all $1 \le p_1, \dots, p_{2l}, r_1, \dots, r_{2l} \le N$ that*

$$\mathrm{E}\{U^{\epsilon_1}_{p_1, r_{\gamma(1)}} \cdots U^{\epsilon_{2l}}_{p_{2l}, r_{\gamma(2l)}}\} = \sum_{\pi \in S_{2l}^{(\epsilon)}} \prod_{k=1}^{l} \delta_{p_k, r_{\gamma(\pi(k))}} \mathrm{Wg}(N, \tilde{\pi}). \tag{5}$$

Proof. Let i_k, i'_k, j_k, j'_k be such that

$$\mathrm{E}\{U^{\epsilon_1}_{p_1, r_{\gamma(1)}} \cdots U^{\epsilon_{2l}}_{p_{2l}, r_{\gamma(2l)}}\} = \mathrm{E}\{U_{i'_1, j'_1} \cdots U_{i'_l, j'_l} U^*_{j_1, i_1} \cdots U^*_{j_l, i_l}\},$$

i.e. $i'_k = r_{p_k}$, $j'_k = r_{\gamma(p_k)}$, $i_k = r_{\gamma(q_k)}$, and $j_k = r_{q_k}$. Thus we have

$$i_k = r_{\gamma(q_k)} = r_{\gamma(\pi(p_{\alpha(k)}))}, \quad \text{and} \quad i'_{\alpha(k)} = r_{p_{\alpha(k)}},$$

and

$$j'_{\beta(k)} = r_{\gamma(p_{\beta(k)})} = r_{\gamma(\pi(q_k))}, \quad \text{and} \quad j_k = r_{q_k}$$

which shows that

$$i_k = i'_{\alpha(k)} \iff r_{p_{\alpha(k)}} = r_{\gamma(\pi(p_{\alpha(k)}))}$$

and

$$j_k = j'_{\beta(k)} \iff r_{q_k} = r_{\gamma(\pi(q_k))}.$$

Thus

$$\prod_{k=1}^{l} \delta_{i_k, i'_{\alpha(k)}} \delta_{j_k, j'_{\beta(k)}} = \prod_{k=1}^{2l} \delta_{r_k, r_{\gamma(\pi(k))}}.$$

Hence

$$\begin{aligned}
\mathrm{E}\{U^{\epsilon_1}_{p_1, r_{\gamma(1)}} \cdots U^{\epsilon_{2l}}_{p_{2l}, r_{\gamma(2l)}}\} &= \mathrm{E}\{U_{i'_1, j'_1} \cdots U_{i'_l, j'_l} U^*_{j_1, i_1} \cdots U^*_{j_l, i_l}\} \\
&= \sum_{\alpha, \beta \in S_n} \delta_{i_1 i'_{\alpha(1)}} \cdots \delta_{i_n i'_{\alpha(n)}} \delta_{j_1 j'_{\beta(1)}} \cdots \delta_{j_n j'_{\beta(n)}} \mathrm{Wg}(\beta\alpha^{-1}) \\
&= \sum_{\pi \in S_{2l}^{(\epsilon)}} \prod_{k=1}^{2l} \delta_{p_k, r_{\gamma(\pi(k))}} \mathrm{Wg}(\tilde{\pi}).
\end{aligned}$$

□

We can now address the question how to calculate expectations of products of traces of our matrices. The following result is exact for each N; later on we will look on its asymptotic version.

Note that the notation $\mathrm{Tr}_\pi[D_1, \dots, D_n]$ for $\pi \in S_n$ is defined in the usual multiplicative way, as was done in Notation 2.4 for φ_1.

Proposition 3.3. *Fix $m_1, \dots, m_k \in \mathbb{N}$ such that $m_1 + \cdots + m_k = 2l$ is even. Let, for fixed $N \in \mathbb{N}$, U be a Haar distributed unitary $N \times N$-random matrix and $D_1, \dots, D_{2l}$ be $N \times N$-random matrices which are independent from U. Let $\epsilon : [2l] \to \{-1, 1\}$ with $\sum_{i=1}^{2l} \epsilon_i = 0$. Put $\gamma = \gamma_{m_1,\dots,m_k}$. Then*

(6)
$$\mathrm{E}\big\{\mathrm{Tr}(D_1 U^{\epsilon_1} \cdots D_{m_1} U^{\epsilon_{m_1}}) \mathrm{Tr}(D_{m_1+1} U^{\epsilon_{m_1+1}} \cdots D_{m_1+m_2} U^{\epsilon_{m_1+m_2}}) \cdots \big\} = \sum_{\pi \in S_{2l}^{(\epsilon)}} \mathrm{Wg}(N, \tilde{\pi}) \cdot \mathrm{E}\big\{\mathrm{Tr}_{\gamma\pi}[D_1, \dots, D_{2l}]\big\}.$$

Proof. Summations over r's and p's in the following formulas are from 1 to N. We denote $\gamma = \gamma_{m_1,\cdots,m_k}$.

$$\begin{aligned}
&\mathrm{E}\big\{\mathrm{Tr}(D_1 U^{\epsilon_1} \cdots D_{m_1} U^{\epsilon_{m_1}}) \mathrm{Tr}(D_{m_1+1} U^{\epsilon_{m_1+1}} \cdots D_{m_1+m_2} U^{\epsilon_{m_1+m_2}}) \cdots \big\} \\
&= \sum_{\substack{r_1,\dots r_{2l} \\ p_1,\dots,p_{2l}}} \mathrm{E}\big\{U^{\epsilon_1}_{p_1, r_{\gamma(1)}} \cdots\ U^{\epsilon_{2l}}_{p_{2l}, r_{\gamma(2l)}}\big\} \cdot \mathrm{E}\big\{(D_1)_{r(1)p(1)} \cdots (D_{2l})_{r(2l)p(2l)}\big\} \\
&= \sum_{\substack{r_1,\dots r_{2l} \\ p_1,\dots,p_{2l}}} \sum_{\pi \in S_{2l}^{(\epsilon)}} \prod_{k=1}^{2l} \delta_{p_k, r_{\gamma(\pi(k))}}\ \mathrm{Wg}(\tilde{\pi}) \cdot \mathrm{E}\big\{(D_1)_{r(1)p(1)} \cdots (D_{2l})_{r(2l)p(2l)}\big\} \\
&= \sum_{\pi \in S_{2l}^{(\epsilon)}} \mathrm{Wg}(\tilde{\pi}) \sum_{\substack{p_1,\dots,p_{2l} \\ r_1,\dots r_{2l}}} \prod_{k=1}^{2l} \delta_{p_k, r_{\gamma(\pi(k))}} \cdot \mathrm{E}\big\{(D_1)_{r(1)p(1)} \cdots (D_{2l})_{r(2l)p(2l)}\big\} \\
&= \sum_{\pi \in S_{2l}^{(\epsilon)}} \mathrm{Wg}(\tilde{\pi}) \mathrm{E}\big\{\mathrm{Tr}_{\gamma\pi}[D_1, \dots, D_{2l}]\big\}.
\end{aligned}$$

□

Motivated by the result of Voiculescu [Voi1, Voi2] that Haar distributed unitary random matrices and constant matrices are asymptotically free, we want to investigate now the corresponding question for second order freeness. It will turn out that one can replace the constant matrices by another ensemble of random matrices, as long as those are independent from the unitary random matrices. Of course, we have to assume that the second ensemble has some asymptotic limit distribution. This is formalized in the following definition. Note that we make a quite strong requirement on the vanishing of the higher order cumulants. This is however in accordance with the observation that in many cases the unnormalized traces converge to Gaussian random variables. Of course, if we have a non-probabilistic ensemble of constant matrices, then the only requirement is the convergence of k_1; all other cumulants are automatically zero.

Definition 3.4. 1) Let $\{A_1, \dots, A_s\}_N$ be a sequence of $N \times N$-random matrices. We say that they have a *second order limit distribution* if there exists a second order non-commutative probability space $(\mathcal{A}, \varphi_1, \varphi_2)$ and $a_1, \dots, a_s \in \mathcal{A}$ such that for all polynomials $p_1, p_2, \dots$ in s non-commuting indeterminates we have

$$\lim_{N\to\infty} \mathrm{k}_1\big\{\mathrm{tr}[p_1(A_1, \dots, A_s)]\big\} = \varphi_1\big(p_1(a_1, \dots, a_s)\big), \tag{7}$$

$$\lim_{N\to\infty} \mathrm{k}_2\big\{\mathrm{Tr}[p_1(A_1, \dots, A_s)], \mathrm{Tr}[p_2(A_1, \dots, A_s)]\big\} = \varphi_2\big(p_1(a_1, \dots, a_s); p_2(a_1, \dots, a_s)\big), \tag{8}$$

and, for $r \geq 3$,

$$\lim_{N\to\infty} \mathrm{k}_r\big\{\mathrm{Tr}[p_1(A_1, \dots, A_s)], \dots, \mathrm{Tr}[p_r(A_1, \dots, A_s)]\big\} = 0. \tag{9}$$

2) We say that two sequences of $N\times N$-random matrices, $\{A_1, \dots, A_s\}_N$ and $\{B_1, \dots, B_t\}_N$, are *asymptotically free of second order* if the sequence $\{A_1, \dots, A_s, B_1, \dots, B_t\}_N$ has a second order limit distribution, given by $(\mathcal{A}, \varphi_1, \varphi_2)$ and $a_1, \dots, a_s, b_1, \dots, b_t \in \mathcal{A}$, and if the unital algebras

$$\mathcal{A}_1 := \mathrm{alg}(1, a_1, \dots, a_s) \qquad \text{and} \qquad \mathcal{A}_2 := \mathrm{alg}(1, b_1, \dots, b_t)$$

are free with respect to (φ_1, φ_2).

Notation 3.5. Fix $m, n \in \mathbb{N}$ and let $\epsilon : [1, m+n] \to \{-1, +1\}$. We defined $S_{m+n}^{(\epsilon)}$ in Notation 3.1, for the case where $\sum_{k=1}^{m+n} \epsilon(k) = 0$, as those permutations in S_{m+n} for which ϵ alternates cyclically between -1 and $+1$ on all cycles. Note that this definition also makes sense in the case where the sum of the ϵ's is not equal to zero, then we just have $S_{m+n}^{(\epsilon)} = \emptyset$. Let ϵ_1 and ϵ_2 be the restrictions of ϵ to $[1, m]$ and to $[m+1, m+n]$, respectively. Then we put

$$S_{NC}^{(\epsilon)}(m, n) := S_{m+n}^{(\epsilon)} \cap S_{NC}(m, n)$$

and

$$NC^{(\epsilon_1)}(m) := S_m^{(\epsilon_1)} \cap NC(m), \qquad NC^{(\epsilon_2)}(n) := S_n^{(\epsilon_2)} \cap NC(n).$$

Theorem 3.6. *Let $\{U\}_N$ be a sequence of Haar distributed unitary $N \times N$-random matrices and $\{A_1, \dots, A_s\}_N$ a sequence of $N \times N$-random matrices which has a second order limit distribution, given by $(\mathcal{A}, \varphi_1, \varphi_2)$ and $a_1, \dots, a_s \in \mathcal{A}$. Furthermore, assume that $\{U\}_N$ and $\{A_1, \dots, A_s\}_N$ are independent. Fix now $m, n \in \mathbb{N}$ and consider polynomials $p_1, \dots, p_{m+n}$ in s non-commuting indeterminates. If we put $(i = 1, \dots, m+n)$*

$$D_i := p_i(A_1, \dots, A_s) \qquad \textit{and} \qquad d_i := p_1(a_1, \dots, a_s),$$

then we have for all $\epsilon(1), \dots, \epsilon(m+n) \in \{-1, +1\}$ that

$$\begin{aligned}
(10)\quad & \lim_{N\to\infty} \mathrm{k}_2\big\{\mathrm{Tr}(D_1U^{\epsilon_1}\cdots D_mU^{\epsilon_m}), \mathrm{Tr}(D_{m+1}U^{\epsilon_{m+1}}\cdots D_{m+n}U^{\epsilon_{m+n}})\big\} \\
& = \sum_{\pi \in S_{NC}^{(\epsilon)}(m,n)} \mu(\tilde{\pi}) \cdot \varphi_1(\gamma_{m,n}\pi)[d_1, \dots, d_{m+n}] \\
& \quad + \sum_{\substack{\pi_1 \in NC^{(\epsilon_1)}(m) \\ \pi_2 \in NC^{(\epsilon_2)}(n)}} \Big(\mu_2(\tilde{\pi}_1, \tilde{\pi}_2) \cdot \varphi_1(\gamma_m\pi_1 \times \gamma_n\pi_2)[d_1, \dots, d_{m+n}] \\
& \qquad\qquad + \mu(\tilde{\pi}_1 \times \tilde{\pi}_2) \cdot \varphi_2(\gamma_m\pi_1, \gamma_n\pi_2)[d_1, \dots, d_{m+n}]\Big).
\end{aligned}$$

Note that in the case where the sum of the ϵ's is different from zero this just states that the limit of k_2 vanishes.

Proof. For notational convenience, we will sometimes write $m+n = 2l$ in the following, and also use $\gamma := \gamma_{m,n}$.

We have

$$\begin{aligned}
& \mathrm{k}_2\big\{\mathrm{Tr}(D_1U^{\epsilon_1}\cdots D_mU^{\epsilon_m}), \mathrm{Tr}(D_{m+1}U^{\epsilon_{m+1}}\cdots D_{2l}U^{\epsilon_{2l}})\big\} \\
& \qquad = \mathrm{E}\big\{\mathrm{Tr}(D_1U^{\epsilon_1}\cdots D_mU^{\epsilon_m})\mathrm{Tr}(D_{m+1}U^{\epsilon_{m+1}}\cdots D_{2l}U^{\epsilon_{2l}})\big\} \\
& \qquad\quad - \mathrm{E}\big\{\mathrm{Tr}(D_1U^{\epsilon_1}\cdots D_mU^{\epsilon_m})\big\} \cdot \mathrm{E}\big\{\mathrm{Tr}(D_{m+1}U^{\epsilon_{m+1}}\cdots D_{2l}U^{\epsilon_{2l}})\big\} \\
& = \sum_{\pi \in S_{2l}^{(\epsilon)}} \mathrm{Wg}(\tilde{\pi}) \cdot \mathrm{E}\big\{\mathrm{Tr}_{\gamma\pi}[D_1, \dots, D_{2l}]\big\} \\
& \quad - \sum_{\substack{\pi_1 \in S_m^{(\epsilon_1)} \\ \pi_2 \in S_n^{(\epsilon_2)}}} \mathrm{Wg}(\tilde{\pi}_1)\mathrm{Wg}(\tilde{\pi}_2) \cdot \mathrm{E}\big\{\mathrm{Tr}_{\gamma_m\pi_1}[D_1, \dots, D_m]\big\} \cdot \mathrm{E}\big\{\mathrm{Tr}_{\gamma_n\pi_2}[D_{m+1}, \dots, D_{2l}]\big\} \\
& \qquad = \sum_{\substack{\pi \in S_{2l}^{(\epsilon)} \\ \pi \text{ connected}}} \mathrm{Wg}(\tilde{\pi}) \cdot \mathrm{E}\big\{\mathrm{Tr}_{\gamma\pi}[D_1, \dots, D_{2l}]\big\}
\end{aligned}$$

$$+\sum_{\substack{\pi_1\in S_m^{(\epsilon_1)}\\ \pi_2\in S_n^{(\epsilon_2)}}}\Big(\mathrm{Wg}(\tilde\pi_1\times\tilde\pi_2)\cdot \mathrm{E}\big\{\mathrm{Tr}_{\gamma_m\pi_1\times\gamma_n\pi_2}[D_1,\dots,D_{2l}]\big\}$$

$$-\mathrm{Wg}(\tilde\pi_1)\mathrm{Wg}(\tilde\pi_2)\cdot\mathrm{E}\big\{\mathrm{Tr}_{\gamma_m\pi_1}[D_1,\dots,D_m]\big\}\cdot\mathrm{E}\big\{\mathrm{Tr}_{\gamma_n\pi_2}[D_{m+1},\dots,D_{2l}]\big\}\Big)$$

The leading order in the first summand for a connected π is given by

$$\mu(\tilde\pi^{-1})N^{-(|\tilde\pi^{-1}|+(m+n)/2)}\cdot N^{\#(\gamma\pi)}\cdot \mathrm{E}\big\{\mathrm{tr}_{\gamma\pi}[D_1,\dots,D_{m+n}]\big\}=$$
$$=N^{m+n-|\pi^{-1}|-|\gamma\pi|}\cdot\mu(\tilde\pi^{-1})\cdot \mathrm{E}\big\{\mathrm{tr}_{\gamma\pi}[D_1,\dots,D_{m+n}]\big\}.$$

Recall that, for a connected π^{-1}, we always have

$$m+n-|\pi^{-1}|-|\gamma\pi|\le 0,$$

and equality is exactly achieved in the case where π^{-1} is annular non-crossing. Thus, in the limit $N\to\infty$ the first sum gives the contribution

$$\sum_{\pi^{-1}\in S_{NC}^{(\epsilon)}(m,n)}\mu(\tilde\pi^{-1})\cdot\varphi_1(\gamma\pi^{-1})[d_1,\dots,d_{m+n}].$$

For a disconnected $\pi_1\times\pi_2$, on the other side, the leading orders in N of all relevant terms are given as follows: $\mathrm{Wg}(\tilde\pi_1\times\tilde\pi_2)$ and $\mathrm{Wg}(\tilde\pi_1)\mathrm{Wg}(\tilde\pi_2)$ have leading order (note that μ is multiplicative)

$$\mu(\tilde\pi_1)\mu(\tilde\pi_2)\left(\frac{1}{N}\right)^{|\pi_1|+|\pi_2|};$$

$\mathrm{E}\big\{\mathrm{Tr}_{\gamma_m\pi_1}[D_1,\dots,D_m]\big\}\cdot\mathrm{E}\big\{\mathrm{Tr}_{\gamma_n\pi_2}[D_{m+1},\dots,D_{m+n}]\big\}$ and $\mathrm{E}\big\{\mathrm{Tr}_{\gamma_m\pi_1\times\gamma_n\pi_2}[D_1,\dots,D_{m+n}]\big\}$ have leading order

$$\varphi_1(\gamma_m\pi_1\times\gamma_n\pi_2)[d_1,\dots,d_{m+n}]\left(\frac{1}{N}\right)^{|\gamma_n\pi_1|+|\gamma_n\pi_2|-(m+n)};$$

$\mathrm{Wg}(\tilde\pi_1\times\tilde\pi_2)-\mathrm{Wg}(\tilde\pi_1)\mathrm{Wg}(\tilde\pi_2)$ has leading order

$$\mu_2(\tilde\pi_1^{-1},\tilde\pi_2^{-1})\cdot\left(\frac{1}{N}\right)^{|\pi_1^{-1}|+|\pi_2^{-1}|-2}$$

and $\mathrm{k}_2\big\{\mathrm{Tr}_{\gamma_m\pi_1}[D_1,\dots,D_m],\mathrm{Tr}_{\gamma_n\pi_2}[D_{m+1},\dots,D_{m+n}]\big\}$ has leading order

$$\varphi_2(\gamma_m\pi_1,\gamma_n\pi_2)[d_1,\dots,d_m;d_{m+1},\dots,d_{m+n}]\left(\frac{1}{N}\right)^{|\gamma_m\pi_1|+|\gamma_n\pi_2|+2-(m+n)}.$$

If we note that

$$|\pi_1^{-1}|+|\gamma_m\pi_1|\ge m+1$$

for all $\pi_1^{-1} \in S_m$, with equality if and only if π_1^{-1} is non-crossing, and the same for π_2^{-1}, then we see that the leading orders in N are coming exactly from non-crossing π_1^{-1} and π_2^{-1} and their contribution is as claimed in the assertion. □

In the following we address the estimates for higher order cumulants, k_r for $r \geq 3$.

For a permutation $\pi \in S_{2l}$, $\epsilon : [1, 2l] \to \{-1, +1\}$ and $N \geq 2l$ we denote

$$\zeta_\pi^{(\epsilon)} = \mathrm{k}_{2l}\big(U_{p_1,q_1}^{\epsilon_1}, \dots, U_{p_{2l},q_{2l}}^{\epsilon_{2l}}\big),$$

where the indices $p_1, \dots, p_{2l}, q_1, \dots, q_{2l}$ were chosen in such a way that

$$\big(p_i = q_j\big) \iff \big(\pi(i) = j\big).$$

It was shown by Collins [C] that

$$\zeta_\pi^{(\epsilon)} = O\big(N^{2-|\pi|-2\#\pi}\big) = O\big(N^{2-2l-\#\pi}\big). \tag{11}$$

If $D_1, \dots, D_l$ are random matrices and $\pi \in S_l$ is a permutation with a cycle structure $\pi = \pi_1 \times \dots \times \pi_r$ with $\pi_i = (\pi_{i,1}, \dots, \pi_{i,l(i)})$ we denote

$$\mathrm{k}_\pi(D_1, \dots, D_l) = \mathrm{k}_r\big(\mathrm{Tr}(D_{\pi_{1,1}} \cdots D_{\pi_{1,l(1)}}), \mathrm{Tr}(D_{\pi_{2,1}} \cdots D_{\pi_{2,l(2)}}), \dots\big).$$

When $\sigma \in S_n$ and $A = \{A_1, \dots, A_k\}$ is a σ–invariant partition of $[1, n]$ we can always write $\sigma = \sigma_1 \times \dots \times \sigma_k$ where σ_i is a permutation of the set A_i. We denote

$$\zeta_{\sigma,A}^{(\epsilon)} = \zeta_{\sigma_1}^{(\epsilon)} \cdots \zeta_{\sigma_k}^{(\epsilon)}$$

and

$$\mathrm{k}_{\sigma,A}(D_1, \dots, D_n) = \mathrm{k}_{\sigma_1}(D_1, \dots, D_n) \cdots \mathrm{k}_{\sigma_k}(D_1, \dots, D_n)$$

by a multiplicative extension. The relation between moments and cumulants implies that for any $\sigma \in S_n$

$$\mathrm{Wg}(\tilde{\sigma}) = \sum_A \zeta_{\sigma,A}^{(\epsilon)},$$

$$\mathrm{E}\big\{\mathrm{Tr}_\sigma(D_1, \dots, D_n)\big\} = \sum_A \mathrm{k}_{\sigma,A}(D_1, \dots, D_n)$$

where the sums run over all σ–invariant partitions A.

Theorem 3.7. *Let $\{U\}_N$ be a sequence of Haar distributed unitary $N \times N$-random matrices and $\{A_1, \dots, A_s\}_N$ a sequence of $N \times N$-random matrices which has a second order limit distribution, given by $(\mathcal{A}, \varphi_1, \varphi_2)$ and $a_1, \dots, a_s \in \mathcal{A}$. Furthermore, assume that $\{U\}_N$ and $\{A_1, \dots, A_s\}_N$ are independent. Fix now k, $m_1, \cdots, m_k \in \mathbb{N}$ and set*

$\gamma=\gamma_{m_1,\dots,m_k}$, $n=m_1+\cdots+m_k$. Consider polynomials $p_1,\dots,p_l$ in s non-commuting indeterminates. We set $(i=1,\dots,l)$

$$D_i := p_i(A_1,\dots,A_s)$$

and consider $\epsilon_1,\dots,\epsilon_n\in\{-1,+1\}$

Then, for every fixed $r\in\mathbb{N}$

(12)

$$\mathrm{k}_r\big\{\mathrm{Tr}(D_1U^{\epsilon_1}\cdots D_{m_1}U^{\epsilon_{m_1}}),\mathrm{Tr}(D_{m_1+1}U^{\epsilon_{m_1+1}}\cdots D_{m_1+m_2}U^{\epsilon_{m_1+m_2}}),\cdots\big\} = \sum_{\pi\in S_n^{(\epsilon)}}\sum_{\substack{A,B\\ A\vee B=1_{[1,n]}}}\zeta^{(\epsilon)}_{\pi,A}\cdot \mathrm{k}_{\gamma\pi,B}(D_1,\dots,D_n),$$

where the second sum runs over pairs (A,B) of partitions of $[1,n]$ such that A is π–invariant and B is $\gamma\pi$–invariant and furthermore $A\vee B=1_{[1,n]}$.

Secondly, we have for $r\geq 3$ that

(13)

$$\lim_{N\to\infty}\mathrm{k}_r\big\{\mathrm{Tr}(D_1U^{\epsilon_1}\cdots D_{m_1}U^{\epsilon_{m_1}}),\mathrm{Tr}(D_{m_1+1}U^{\epsilon_{m_1+1}}\cdots D_{m_1+m_2}U^{\epsilon_{m_1+m_2}}),\cdots\big\}=0.$$

Proof. In order to show (12) it is enough to use Proposition 3.3 and to see that (12) indeed fulfills the defining property of cumulants.

In order to show (13), we have to control the order of the appearing products $\zeta^{(\epsilon)}_{\pi,A}\cdot\mathrm{k}_{\gamma\pi,B}$.

Let c_i denote the number of blocks of B which contain exactly i cycles of $\gamma\pi$. By the definition of these quantities we have, by using (11), that

$$\zeta^{(\epsilon)}_{\pi,A}=O\big(N^{2\#A-n-\#\pi}\big)$$

and, by using our assumption on the limit distribution of the D's, that

$$\mathrm{k}_{\gamma\pi,B}=\begin{cases}O\big(N^{c_1}\big), & \text{if } B \text{ has only blocks of size 1 and 2}\\ o\big(N^{c_1}\big), & \text{if } B \text{ has at least one block of size} \geq 3\end{cases}$$

Note that

$$c_1=\#(\gamma\pi)-\sum_{i\geq 2} ic_i.$$

Thus we get

$$\zeta^{(\epsilon)}_{\pi,A}\cdot\mathrm{k}_{\gamma\pi,B}(D_1,\dots,D_n)=\begin{cases}O\big(N^{2\#A-n-\#\pi+\#(\gamma\pi)-2c_2}\big) & \text{if } c_3+c_4+\cdots=0,\\ o\big(N^{2\#A-n-\#\pi+\#(\gamma\pi)-\sum_{i\geq2}ic_i}\big) & \text{if } c_3+c_4+\cdots\geq 1.\end{cases}$$

Suppose first that $c_3+c_4+\cdots\geq 1$; then

$$\sum_{i\geq 2} ic_i=(c_2+c_3+\cdots)+\sum_{i\geq 1}(i-1)c_i\geq 1+(\#(\gamma\pi)-\#B)$$

and hence

$$\zeta^{(\epsilon)}_{\pi,A} \cdot \mathrm{k}_{\gamma\pi,B}(D_1,\dots,D_l) = o\big(N^{2\#A-n-\#\pi+\#(\gamma\pi)-1-\#(\gamma\pi)+\#B}\big).$$

Note now that the requirement $A \vee B = 1_{[1,n]}$ implies that

$$\#A + \#B \leq n+1. \tag{14}$$

So we can in this case estimate our asymptotics against

$$o\big(N^{\#A-\#\pi}\big),$$

which goes to zero in any case, because $\#A \leq \#\pi$.

Suppose now, on the other hand, that $c_3 + c_4 + \cdots = 0$; then

$$\#(\gamma\pi) - \#B = c_2,$$

and thus

$$\zeta^{(\epsilon)}_{\pi,A} \cdot \mathrm{k}_{\gamma\pi,B}(D_1,\dots,D_n) = O\big(N^{2\#A-n-\#\pi+\#(\gamma\pi)-2(\#(\gamma\pi)-\#B)}\big).$$

Using again (14) and

$$\#\pi + \#(\gamma\pi) = 2n - (|\pi| + |\gamma\pi|) \geq 2n - |\gamma| = n + r$$

we can estimate the asymptotics in this case against

$$O\big(N^{2-r}\big),$$

which gives, for $r \geq 3$, the required bound. □

Theorem 3.8. *Let $\{U\}_N$ be a sequence of unitary $N \times N$-random matrices and $\{A_1,\dots,A_s\}_N$ a sequence of $N \times N$-random matrices which has a second order limit distribution. If $\{U\}_N$ and $\{A_1,\dots,A_s\}_N$ are independent, then they are asymptotically free of second order.*

Proof. The asymptotic freeness with respect to $\mathrm{k}_1\{\mathrm{tr}[\cdot]\}$ is essentially the same argument as Voiculescu's proof [Voi1, Voi2] for the case of constant matrices, see also the proof of Collins [C].

Theorem 3.7 provides the bound on higher order cumulants so we need to prove now only the second order statement.

We have to consider cyclically alternating and centered words in the U's and the A's. For the U's, every centered word is a linear combination of non-trivial powers of U, thus it suffices to consider such powers. Thus we have to look at expressions of the form

$$\mathrm{k}_2\big\{\mathrm{Tr}(B_1U^{i(1)}\cdots B_pU^{i(p)}), \mathrm{Tr}(U^{j(r)}C_r\cdots U^{j(1)}C_1)\big\}, \tag{15}$$

where the B's and the C's are centered polynomials in the A's and $i(1),\dots,i(p),j(1),\dots,j(r)$ are integers different from zero. We have to show that in the limit $N \to \infty$ the expression (15) converges to (16)

$$\delta_{pr} \sum_{k=0}^{p-1} \varphi_1(B_1 C_{1+k}) \varphi_1(U^{i(1)} U^{j(1+k)}) \cdots \varphi_1(B_p C_{p+k}) \varphi_1(U^{i(p)} U^{j(p+k)}).$$

We can bring the expression (15) into the form considered in Theorem 3.6 by inserting 1's between neighbouring factors U or neighbouring factors U^*. If we relabel the B's, C's, and 1's as D's then we have to look at the following situation: For polynomials p_i in s non-commuting indeterminates we consider

$$D_i := p_i(A_1, \dots, A_s),$$

which are either asymptotically centered or equal to 1. The latter case can only appear if we have cyclically the pattern $\dots U D_i U \dots$ or $\dots U^* D_i U^* \dots$. Formally, this means:

- if $\epsilon_{\gamma^{-1}(i)} = \epsilon_i$ then either $D_i = 1$ (for all N, i.e., $p_i = 1$) or
$$\lim_{N \to \infty} \mathrm{k}_1\big\{\mathrm{tr}[D_i]\big\} = 0.$$
- if $\epsilon_{\gamma^{-1}(i)} \neq \epsilon_i$ then
$$\lim_{N \to \infty} \mathrm{k}_1\big\{\mathrm{tr}[D_i]\big\} = 0.$$

We can now use Theorem 3.6 for calculating the limit

$$\lim_{N \to \infty} \mathrm{k}_2\big\{\mathrm{Tr}(D_1 U^{\epsilon_1} \cdots D_m U^{\epsilon_m}), \mathrm{Tr}(D_{m+1} U^{\epsilon_{m+1}} \cdots D_{m+n} U^{\epsilon_{m+n}})\big\},$$

and we will argue that most terms appearing there will vanish. Consider first the last two sums, corresponding to $\pi_1 \in NC(m)$ and $\pi_2 \in NC(n)$. Since π_1 is non-crossing we have that $\#\pi_1 + \#(\gamma_m \pi_1^{-1}) = m+1$. Since each cycle of π_1 must contain at least one U and one U^*, we have

$$\#\pi_1 \leq \frac{m}{2},$$

which implies $\#(\gamma_m \pi_1^{-1}) \geq m/2 + 1$. However, this can only be true if $\gamma_m \pi_1^{-1}$ contains at least two singletons. Note that if (i) is a singleton of $\gamma_m \pi_1^{-1}$ and if we have $D_i = 1$ for that i, then we have

$$\gamma_m \pi_1^{-1}(i) = i, \qquad \text{thus} \qquad \pi_1^{-1}(i) = \gamma_m^{-1}(i) = \gamma^{-1}(i),$$

and hence

$$\epsilon_{\pi_1^{-1}(i)} = \epsilon_{\gamma^{-1}(i)} = \epsilon_i,$$

which is not allowed because π_1 is from $NC^{(\epsilon_1)}(m)$, i.e., it must connect alternatingly U with U^*. Thus, both

$$\varphi_1(\gamma_m \pi_1 \times \gamma_n \pi_2)[d_1, \dots, d_{m+n}]$$

and

$$\varphi_2(\gamma_m\pi_1, \gamma_n\pi_2)[d_1, \dots, d_{m+n}]$$

are zero, because at least one singleton (i) gives the contribution $\varphi_1(d_i) = 0$.

Consider now the first summand, for a $\pi \in S_{NC}^{(\epsilon)}(m, n)$. Let us put again $\gamma := \gamma_{m,n}$. Since π is annular non-crossing we have

$$|\pi| + |\gamma\pi^{-1}| = m + n,$$

or

$$\#\pi + \#(\gamma\pi^{-1}) = m + n.$$

Again, each cycle of π must contain at least two elements, i.e.,

$$\#\pi \leq \frac{m+n}{2},$$

thus

$$\#(\gamma\pi^{-1}) \geq \frac{m+n}{2}.$$

If $\gamma\pi^{-1}$ has a singleton (i), then this will contribute $\varphi_1(d_i)$ and since, as above the case $d_i = 1$ is excluded for a singleton, we get a vanishing contribution in this case. This implies that, in order to get a non-vanishing contribution, $\gamma\pi^{-1}$ must contain no singletons, which, however, means that we must have

$$\#(\gamma\pi^{-1}) = \frac{m+n}{2}, \qquad \text{and thus also} \qquad \#\pi = \frac{m+n}{2}$$

i.e., all cycles of $\gamma\pi^{-1}$ and of π contain exactly two elements. This, however, can only be the case if each cycle connects the outer circle with the inner circle. Being non-crossing fixes the permutation up to a rotation of the inner circle. Thus, in order to get a non-vanishing contribution, we need $m = n$ and

$$\pi = (1, \gamma^k(2n))(2, \gamma^k(2n-1)), \dots, (n, \gamma^k(n+1))$$

for some $k = 0, 1, \dots, n-1$. Note that π must always couple a U with a U^* and the factor $\mu(\tilde{\pi})$ is always 1 for such pairings. This gives exactly the contribution as needed for second order freeness. $\square$

Let us exploit a bit more the implications of Theorem 3.6. In particular, we can choose there all D_i equal to 1. Then we have that all φ_1 contribute a factor 1 and all φ_2 contribute a factor 0. Thus the third term in Eq. (10) vanishes and we get the following formula for the limit of k_2.

Corollary 3.9. *Let $\{U\}_N$ be a sequence of Haar distributed unitary $N \times N$-random matrices. Then $\{U\}_N$ has a second order limit distribution which is given by*

$$\lim_{N\to\infty} \mathrm{k}_2\big\{\mathrm{Tr}(U^{\epsilon_1}\cdots U^{\epsilon_m}), \mathrm{Tr}(U^{\epsilon_{m+1}}\cdots U^{\epsilon_{m+n}})\big\} = \sum_{\pi\in S_{NC}^{(\epsilon)}(m,n)} \mu(\tilde{\pi}) + \sum_{\substack{\pi_1\in NC^{(\epsilon_1)}(m)\\ \pi_2\in NC^{(\epsilon_2}(n)}} \mu_2(\tilde{\pi}_1,\tilde{\pi}_2) \tag{17}$$

Since U is unitary, we can reduce the considered products of U and U^* either to 1, a power of U or a power of U^*. In this reduced form the above corollary recovers a classical result of Diaconis and Shahshahani [DS]. (One should, however, note that Corollary 3.9 has also some merits in its general non-reduced form. In principle, it allows to derive the values of μ_2. These kind of questions will be considered elsewhere.)

Corollary 3.10. *Let $\{U\}_N$ be a sequence of Haar distributed unitary $N \times N$-random matrices. Then $\{U\}_N$ has a second order limit distribution, which is given by ($m, n \geq 0$)*

$$\lim_{N\to\infty} \mathrm{k}_2\big\{\mathrm{Tr}(U^m), \mathrm{Tr}(U^n)\big\} = 0 \tag{18}$$

and

$$\lim_{N\to\infty} \mathrm{k}_2\big\{\mathrm{Tr}(U^m), \mathrm{Tr}(U^{*n})\big\} = n\delta_{mn} \tag{19}$$

Proof. The main observation to be made is that contributing permutations must connect alternatingly a U with a U^*. Thus, in the case of $\mathrm{k}_2\big\{\mathrm{Tr}(U^m), \mathrm{Tr}(U^n)\big\}$ there are no contributing permutations at all and we get zero in this case. In the other case, there are no possibilities for π_1 or π_2 and the only $\pi \in S_{NC}^{(\epsilon)}(m,n)$ which connect in this alternating way are pairings, where each block must contain one U and one U^*. This forces m and n to be equal. In that case there are n possibilities for such pairings: we have the freedom of pairing the first U with any of the U^*. After this choice is made the rest is determined. Since $\mu(\tilde{\pi})$ is always 1 for such pairings we get the claimed formula. □

Of course, a natural question in this context is how the above result generalizes to the case of several independent unitary random matrices. Note that after we have established the existence of a second order limit distribution for Haar distributed unitary random matrices we can use an independent copy of them as the ensemble $\{A_1, \ldots, A_s\}$ in our Theorem 3.8. Clearly this can be iterated to give the following.

Theorem 3.11. *Let $\{U^{(1)}\}_N, \dots, \{U^{(r)}\}_N$ be r sequences of Haar distributed unitary $N \times N$-random matrices. If $\{U^{(1)}\}_N, \dots, \{U^{(r)}\}_N$ are independent, then they are asymptotically free of second order.*

This contains the information about the fluctuation of several independent Haar distributed unitary random matrices. Again, it suffices to consider traces of reduced words in our random matrices, i.e., expressions of the form

$$\operatorname{Tr}[U_{i(1)}^{k(1)} \cdots U_{i(n)}^{k(n)}] \tag{20}$$

for $n \in \mathbb{N}$, and $k(r) \in \mathbb{Z} \backslash \{0\}$ and $i(r) \neq i(r+1)$ for all $r = 1, \dots, n$ (where $i(n+1) = i(1)$). But these are now products in cyclically alternating and centered variables, so that by the very definition of second order freeness we get

$$\begin{aligned} &\lim_{N\to\infty} \mathrm{k}_2 \big\{ \operatorname{Tr}[U_{i(1)}^{k(1)} \cdots U_{i(m)}^{k(m)}], \operatorname{Tr}[U_{j(n)}^{l(n)} \cdots U_{j(1)}^{l(1)}] \big\} \\ &\qquad = \delta_{mn} \sum_{r=0}^{n-1} \varphi_1 \big(U_{i(1)}^{k(1)} U_{i(1+r)}^{k(1+r)} \big) \cdots \varphi_1 \big(U_{i(n)}^{k(n)} U_{i(n+r)}^{k(n+r)} \big). \end{aligned} \tag{21}$$

The contribution of φ_1 in these terms vanishes unless the matrices and their powers match. Note also that the vanishing of higher cumulants can be rephrased in a more probabilistic language by saying that the random variables (20) converge to a Gaussian family.

Corollary 3.12. *Let $\{U_{(1)}\}_N, \dots, \{U_{(r)}\}_N$ be independent sequences of Haar distributed unitary $N \times N$-random matrices. Then, the collection (20) of unnormalized traces in cyclically reduced words in these random matrices converges to a Gaussian family of centered random variables whose covariance is given by the number of matchings between the two reduced words,*

$$\begin{aligned} &\lim_{N\to\infty} \mathrm{k}_2 \big\{ \operatorname{Tr}[U_{i(1)}^{k(1)} \cdots U_{i(m)}^{k(m)}], \operatorname{Tr}[U_{j(n)}^{l(n)} \cdots U_{j(1)}^{l(1)}] \big\} \\ &\qquad = \delta_{mn} \cdot \# \big\{ r \in \{1, \dots, n\} \mid i(s) = j(s+r), \\ &\qquad\qquad\qquad k(s) = -l(s+r) \quad \forall s = 1, \dots, n \big\} \end{aligned} \tag{22}$$

This result was also obtained independently in the recent work of Rădulescu [R] around Connes's embedding problem.

The following theorem gives an easy way to construct families of random matrices which are asymptotically free of second order.

Theorem 3.13. *Let $\{U\}_N$ be a sequence of Haar distributed unitary $N \times N$-random matrices, let $\{A_1, \dots, A_s\}_N$ be a sequence of $N \times N$-random matrices which has a second order limit distribution and let*

$\{B_1, \ldots, B_t\}_N$ be another sequence of $N \times N$-random matrices which has a second order limit distribution. Furthermore, assume that $\{A_1, \ldots, A_s, B_1, \ldots, B_t\}_N$ and $\{U\}_N$ are independent. Then the sequences $\{A_1, \ldots, A_s\}_N$ and $\{UB_1U^{-1}, \ldots, UB_tU^{-1}\}_N$ are asymptotically free of second order.

Proof. Observe that under the additional assumption that $\{A_1, \ldots, A_s, B_1, \ldots, B_t\}$ has a second order limit distribution it is enough to apply Theorem 3.8. In order to show that the latter assumption is not necessary, one has to revisit the proof to see that trivial bounds on mixed moments and cumulants are sufficient to show asymptotic second order freeness.

□

We say that a tuple $\{B_1, \ldots, B_s\}$ of $N \times N$-random matrices is $\mathcal{U}(N)$–invariant if for every $U \in \mathcal{U}(N)$ the joint probability distribution of the random matrices $\{B_1, \ldots, B_s\}$ coincides with the joint probability distribution of the random matrices $\{UB_1U^{-1}, \ldots, UB_sU^{-1}\}$.

Corollary 3.14. *Let $\{A_1, \ldots, A_s\}_N$ be a sequence of $N \times N$-random matrices which has a second order limit distribution and let $\{B_1, \ldots, B_t\}_N$ be a sequence of $\mathcal{U}(N)$–invariant $N \times N$-random matrices which has a second order limit distribution. Furthermore assume that the matrices $\{A_1, \ldots, A_s\}_N$ and the matrices $\{B_1, \ldots, B_t\}_N$ are independent. Then the sequences $\{A_1, \ldots, A_s\}_N$ and $\{B_1, \ldots, B_t\}_N$ are asymptotically free of second order.*

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