Characters of symmetric groups and free cumulants

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Abstract. We investigate Kerov's formula expressing the normalized irreducible characters of symmetric groups evaluated on a cycle, in terms of the free cumulants of the associated Young diagrams.

1. Introduction

Let μ be a probability measure on $\mathbb{R}$, with compact support. Its Cauchy transform has the expansion

$$G_\mu(z) = \int_{\mathbb{R}} \frac{1}{z-x}\mu(dx) = z^{-1} + \sum_{k=1}^{\infty} M_k\, z^{-k-1} \tag{1.1}$$

where the M_k are the moments of the measure μ. This Laurent series has an inverse for composition $K(z)$, with an expansion

$$K_\mu(z) = z^{-1} + \sum_{k=1}^{\infty} R_k z^{k-1}. \tag{1.2}$$

The R_k are called the free cumulants of μ and can be expressed as polynomials in terms of the moments. Free cumulants show up in the asymptotic behaviour of characters of large symmetric groups. More precisely, let λ be a Young diagram, to which we associate a piecewise affine function $\omega : \mathbb{R} \to \mathbb{R}$, with slopes ± 1, such that $\omega(x) = |x|$ for $|x|$ large enough, as in Fig. 1 below, which corresponds to the partition $8 = 4 + 3 + 1$. Alternatively we can encode the Young diagram using the local minima and local maxima of the function ω, denoted by $x_1, \dots, x_m$ and $y_1, \dots, y_{m-1}$ respectively, which form two interlacing sequences of integers. These

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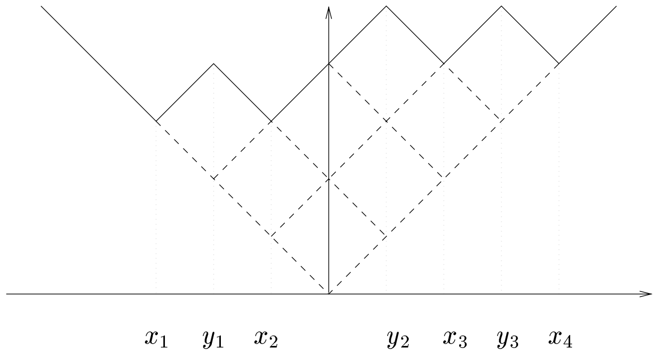
are (-3,-1,2,4) and (-2,1,3) respectively in the picture.



$Fig.1$

Associated with the Young diagram there is a unique probability measure μ_ω on the real line, such that

$$\int_{\mathbb{R}} \frac{1}{z-x}\mu_\omega(dx) = \frac{\prod_{i=1}^{m-1}(z-y_i)}{\prod_{i=1}^{m}(z-x_i)} \quad \text{for all } z \in \mathbb{C}\setminus\mathbb{R}. \tag{1.3}$$

This probability measure is supported by the set $\{x_1, \dots, x_k\}$ and is called the transition measure of the diagram, see [**K1**]. We shall denote by $R_j(\omega)$ its free cumulants. Let $\sigma \in S_n$ be a permutation with k_2 cycles of length 2, k_3 of length 3, etc. We shall keep $k_2, k_3, ...$ fixed, and denote $r = \sum_{j=2}^{\infty} jk_j$, while we let $n \to \infty$. The normalized character χ_ω associated to a Young diagram with n cells has the following asymptotic evaluation from [**B**]

$$\chi_\omega(\sigma) = \prod_{j=2}^{\infty} R_{j+1}^{k_j}(\omega) n^{-r} + O(n^{-\frac{r+1}{2}}). \tag{1.4}$$

Here the O term is uniform over all Young diagrams whose numbers of rows and columns are $\leq A\sqrt{n}$ for some constant A, and all permutations with $r \leq r_0$ for some r_0.

As remarked by S. Kerov [**K2**], free cumulants can be used to get universal, exact formulas for character values. More precisely consider the following quantities

$$\Sigma_k(\omega) = n(n-1)\dots(n-k+1)\chi_\omega(c_k)$$

for $k \geq 1$ where c_k is a cycle of order k (with $c_1 = e$).

Theorem 1.1 (Kerov's formula for characters). *There exist universal polynomials $K_1, K_2, \dots, K_m, \dots,$ with integer coefficients, such that the following identities hold for any n and any Young diagram ω with n cells*

$$\Sigma_k(\omega) = K_k(R_2(\omega), R_3(\omega), \dots, R_{k+1}(\omega)).$$

We list the few first such polynomials

$$\begin{aligned}
&\Sigma_1 = R_2\\
&\Sigma_2 = R_3\\
&\Sigma_3 = R_4 + R_2\\
&\Sigma_4 = R_5 + 5R_3\\
&\Sigma_5 = R_6 + 15R_4 + 5R_2^2 + 8R_2\\
&\Sigma_6 = R_7 + 35R_5 + 35R_3R_2 + 84R_3
\end{aligned}$$

The coefficients of Kerov's polynomials seem to have some interesting combinatorial significance, although the situation is far from being understood. In this paper we shall give a proof of the above theorem, compute some of the coefficients in the formula, as well as give some insight into this problem.

This paper is organized as follows. In Section 2 we gather some information on free cumulants, Boolean cumulants and their combinatorial significance. In Section 3 we introduce some elements in the center of the symmetric group algebra. These are used in Section 4 to give a combinatorial proof of Theorem 1.1. In Section 5 we give another proof, based on a formula of Frobenius, which yields a computationally efficient formula for computing Kerov's polynomials. In Section 6 we compute the coefficients of the linear terms of Kerov's polynomials, as well as some coefficients of degree 2. We make some remarks in Section 7 on the possible combinatorial significance of the coefficients of Kerov's polynomials. This involves in a natural way the Cayley graph of the symmetric group. Finally in Section 8 we list the values of Kerov polynomials up to Σ_{11}.

I would like to thank A. Okounkov and R. Stanley for useful communication, as well as G. Olshanski for providing me a copy of [**IO**].

2. Noncrossing partitions, moments and free cumulants

From the relation between moments and cumulants given by

$$G_\omega = K_\omega^{\langle -1\rangle} \tag{2.1}$$

we obtain by Lagrange inversion formula that

$$R_k = -\frac{1}{k-1}[z^{-1}]\,G_\omega(z)^{-k+1} \tag{2.2}$$

(where $[z^{-1}]\,L(z)$ denotes the coefficient of z^{-1} in the expansion of a Laurent series $L(z)$). From this we get that the coefficient of $M_1^{l_1}\dots M_r^{l_r}$ in R_k is equal to

$$(-1)^{1+l_1+\dots+l_r}\frac{(k-2+\sum_i l_i)!}{l_1!\dots l_r!(k-1)!}. \tag{2.3}$$

if $k=\sum_j jl_j$, and to 0 if not.

Conversely one has

$$M_k = \frac{1}{k+1}[z^{-1}]\,K(z)^{k+1}$$

and the coefficient of $R_1^{l_1}\dots R_r^{l_r}$ in M_k, with $k=\sum_i il_i$ is equal to

$$\frac{k!}{l_1!\dots l_r!(k+1-\sum_i l_i)!} \tag{2.4}$$

It will be also interesting to introduce the series

$$H_\omega(z) = 1/G_\omega(z) = z - \sum_{k=1}^{\infty} B_k z^{1-k}$$

The coefficients B_k in this formula are called Boolean cumulants [**SW**] and the coefficient for $B_1^{l_1}\dots B_r^{l_r}$ in M_k, with $\sum_j jl_j = k$ is the multinomial coefficient

$$\frac{(l_1+l_2+\dots+l_r)!}{l_1!l_2!\dots l_r!} \tag{2.5}$$

A combinatorial interpretation of these formulas is afforded by R. Speicher's work [**Sp**] which we recall now. A noncrossing partition of $\{1,\dots,k\}$ is a partition such that there are no a, b, c, d with $a < b < c < d$, a and c belong to some block of the partition and c, d belong to some other block. The noncrossing partitions form a ranked lattice which will be denoted by $NC(k)$. We shall use the order opposite to the refinement order so that the rank of a non-crossing partition is $k+1-$(number of parts). The relation between moments and free cumulants now reads

$$M_k = \sum_{\pi \in NC(k)} R[\pi] \tag{2.6}$$

where, for a noncrossing partition $\pi = (\pi_1, \pi_2, \dots \pi_r)$, one has $R[\pi] = \prod_i R_{|\pi_i|}$, where $|\pi_i|$ is the number of elements of the part π_i. It follows from (2.5) that the coefficient of $R_1^{l_1} \dots R_r^{l_r}$ in the expression of M_k (with $k = \sum i l_i$) is equal to the number of non-crossing partitions in $NC(k)$ with l_i parts of i elements, and is given by (2.4).

A parallel development can be made for the connection between Boolean cumulants and moments. A partition of $\{1, \dots, k\}$ is called an interval partition if its parts are intervals. The interval partitions form a lattice $B(k)$, which is isomorphic to the lattice of all subsets of $\{1, \dots, k-1\}$ (assign to an interval partition the complement in $\{1, \dots, k\}$ of the set of largest elements in the parts of the partition). The formula for expressing the M_k in terms of the B_k is

$$M_k = \sum_{\pi \in B(k)} B[\pi] \tag{2.7}$$

3. On central elements in the group algebra of the symmetric group

Let λ be a Young diagram, and for $n \geq |\lambda|$ let ϕ be a one to one map from the cells of λ to the set $\{1, \dots, n\}$. Consider the associated permutation σ_ϕ whose cycles are given by the rows of the map ϕ. For example the following map with $n \geq 24$, gives the permutation with cycle decomposition

(4 7 11 2 19 12)(20 21 1 14 24 16)(5 3)(22 13)(2 9)

4	7	11	2	19	12
20	21	1	14	24	16
5	3				
22	13				
2	9				
10					
8					

Fig.2

If Φ_λ is the set of such maps defined on λ, we shall call $a_{\lambda;n}$ the element in the group algebra of S_n given by

$$a_{\lambda;n} = \sum_{\phi \in \Phi_\lambda} \sigma_\phi$$

see [**KO**]. If λ has one row, of length l, we call $a_{l;n}$ the corresponding element. Note that $a_{1;n} = n.e$.

LEMMA 3.1. *There exists universal polynomials P_λ with integer coefficients such that, for all n, one has*

$$a_{\lambda;n} = P_\lambda(a_{1;n}, \dots, a_{|\lambda|;n})$$

and $\sum_j j \deg_{P_\lambda}(a_{j;n}) \leq |\lambda|$.

The proof is by induction on the number of cells of λ. This is clear by definition if λ has one row. If λ has more than one row then let λ' be λ with the last row deleted, and let k be the length of this row. One has

$$a_{\lambda';n}a_{k;n} = \sum_{\phi_1,\phi_2} \sigma_{\phi_1}\sigma_{\phi_2}$$

where ϕ_2 is a map on the diagram λ' and ϕ_2 a map on the diagram (k) with one row of length k. For any pair (ϕ_1, ϕ_2) there is a unique pair A, B where A is a subset of cells of λ', B is a subset of cells of (k), and a bijection τ from A to B which tells on which cells the two maps ϕ_1 and ϕ_2 coincide. The cycle structure of $\sigma_{\phi_1}\sigma_{\phi_2}$ depends only on λ', k, A, B, τ, and not on the values taken by the maps ϕ_1, ϕ_2. Let $\Lambda_{\lambda',k,A,B,\tau}$ be the diagram with $|\lambda| - |A|$ boxes (putting some one-box rows if necessary) of this conjugacy class. For each (A, B, τ) take some corresponding (ϕ_1, ϕ_2), and take a map on the diagram $\Lambda_{\lambda',k,A,B,\tau}$, which realizes the permutation $\sigma_{\phi_1}\sigma_{\phi_2}$. If necessary put the fixed points in the one-box rows. Now extend this to all pairs of maps (ϕ_1, ϕ_2) covariantly with respect to the action of S_n. Each map on $\Lambda_{\lambda',k,A,B,\tau}$ is obtained exactly once, and if $A = B = \emptyset$ then clearly $\Lambda_{\lambda',k,A,B,\tau} = \lambda$. It follows that

$$a_{\lambda';n}a_{k;n} = a_{\lambda;n} + \sum_{A,B,\tau,|A|\geq 1} a_{\Lambda_{\lambda',A,B,\tau};n} \tag{3.1}$$

For all terms in the sum one has $|\Lambda_{\lambda',A,B,\tau}| < |\lambda|$. The proof follows by induction. The condition on degrees is checked also by induction. $\square$

4. Jucys-Murphy elements and Kerov's formula

Consider the symmetric group S_n acting on $\{1, 2, \dots, n\}$ and let $*$ be a new symbol. We imbed S_n into S_{n+1} acting on $\{1, 2, \dots, n\} \cup \{*\}$. In the group algebra $\mathbb{C}(S_{n+1})$, consider the Jucys-Murphy element

$$J_n = (1\,*) + (2\,*) + \ldots + (n\,*)$$

where $(i\,j)$ denotes the transposition exchanging i and j. Let E_n denote the orthogonal projection from $\mathbb{C}(S_{n+1})$ onto $\mathbb{C}(S_n)$, i.e. $E_n(\sigma) = \sigma$ if $\sigma \in S_n$, and $E_n(\sigma) = 0$ if not. If we endow $\mathbb{C}(S_{n+1})$ with its canonical trace $\tau(\sigma) = \delta_{e\sigma}$ (i.e. τ is the linear extension of the normalized character of the regular representation), then E_n is the conditional expectation onto $\mathbb{C}(S_n)$, with respect to τ. We define the moments of the Jucys-Murpy elements by

$$\mathcal{M}_k = E_n(J_n^k) \tag{4.1}$$

To this sequence of moments we can associate a sequence of free cumulants through the construction of section 2. We call $\mathcal{R}_k$ these free cumulants. By construction, the $\mathcal{M}_k$ and $\mathcal{R}_k$ belong to the center of the group algebra $\mathbb{C}(S_n)$ (even to $\mathbb{Z}(S_n)$). The relevance of these moments and cumulants is the following

LEMMA 4.1. *For any n and any Young diagram ω with n boxes, one has*

$$\chi_\omega(\mathcal{R}_k) = R_k(\omega)$$

Since χ_ω is an irreducible character, it is multiplicative on the center of the symmetric group algebra and therefore it is enough to check that $\chi_\omega(\mathcal{M}_k) = M_k(\omega)$. Let χ^*_ω be the induced character on S_{n+1}, then $\chi_\omega(\mathcal{M}_k) = \chi^*_\omega(J^k_n)$ and the result follows from the computation of eigenvalues of Jucys-Murphy elements. See e.g. [**B**], Section 3. □

One has

$$J^k_n = \sum_{i_1,\dots,i_k\in\{1,\dots,n\}} (*\, i_1)\dots(*\, i_k) \tag{4.2}$$

A term in this sum gives a non trivial contribution to $\mathcal{M}_k$ if and only if the permutation $\sigma = (*\, i_1)\dots(*\, i_k)$ fixes $*$. In order to see when this happens we have to follow the images of $*$ by the successive partial products of transpositions. Let $j_1 = \sup\{l < k \,|\, i_l = i_k\}$. If this set is empty then $\sigma(*) = i_k \neq *$. If not then one has $\sigma = (*i_1)\dots(*i_{j-1})\sigma'$ where $\sigma'(*) = *$, hence we can continue and look for $j_2 = \sup\{l < j_1 - 1 \,|\, i_l = i_{j_1-1}\}$. In this way we construct a sequence $j_1, j_2, \dots$. If, and only if, the last term of this sequence is 1 then we get a non trivial contribution. Let π be the partition of $1,\dots,k$ such that l and m belong to the same part if and only if $i_l = i_m$. The fact that $(*\, i_1)\dots(*\, i_k)$ fixes $*$ depends only on this partition, and we call admissible partitions the ones for which $(*\, i_1)\dots(*\, i_k)$ fixes $*$. Furthermore, the conjugacy class, in S_n, of $(*\, i_1)\dots(*\, i_k)$ depends only on the partition. Let $\lambda(\pi)$ be the Young diagram formed with the nontrivial cycles of this conjugacy class. Let

$$\mathcal{Z}_\pi = \sum_{i_1,\dots,i_k\sim\pi} (*\, i_1)\dots(*\, i_k)$$

where $i_1,\dots,i_k \sim \pi$ means that the partition associated to the sequence $i_1,\dots,i_k$ is π, then we have

$$\mathcal{M}_k = \sum_{\pi\ \text{admissible}} \mathcal{Z}_\pi$$

Let $c(\pi)$ be the number of parts of π, then the number of k-tuples $i_1,\dots,i_k \sim \pi$ is equal to $(n)_{c(\pi)}$ (where, as usual $(n)_k = n(n-1)\dots(n-k+1)$), and one has $|\lambda(\pi)| \le c(\pi)$, therefore one has

$$\mathcal{Z}_\pi = \frac{(n)_{c(\pi)}}{(n)_{|\lambda|}} a_{\lambda(\pi);n} = (n-|\lambda|)\dots(n-c(\pi)+1)a_{\lambda(\pi);n}$$

In order that $\lambda(\pi)$ be a cycle of length $k-1$, it is necessary an sufficient that π be the partition $\{1,k\},\{2\},\{3\},\dots,\{k-1\}$. All other admissible partitions have $c(\pi) < k-1$. We deduce that

$$\mathcal{M}_k = a_{k-1;n} + \sum_{\pi\ \ \text{admissible},\, c(\pi)<k-1} Z_\pi$$

It follows from Section 3 that $\mathcal{M}_k$ is a polynomial with integer coefficients, independent of n, in the $a_{j;n}$, of the form

$$a_{k-1;n} + (\text{polynomial in } a_{j;n}; j < k-1).$$

We can thus invert this polynomial relation and get

$$a_{k-1;n} = \mathcal{M}_k + (\text{polynomial in } \mathcal{M}_j; j < k).$$

with polynomial with integer coefficients. Since $\mathcal{M}_k$ can be expanded as polynomials with integer coefficients in the $\mathcal{R}_j$ we thus have

$$a_{k-1;n} = \mathcal{R}_k + (\text{polynomial in } \mathcal{R}_j; j < k).$$

Applying χ_ω to both sides of this equation and using Lemma 4.1, we obtain Theorem 1.1. □

5. Frobenius formula and free cumulants

Let $\lambda = (\lambda_1, \lambda_2, \dots)$ be a partition of n, with $\lambda_1 \geq \lambda_2 \geq \dots$, and $\mu_i = \lambda_i + n - i$. Let

$$\varphi(z) = \prod (z - \mu_i),$$

then the value of the normalized character χ_λ on a cycle of length k is given by Frobenius' formula

$$(n)_k \chi_\lambda(c_k) = -\frac{1}{k}[z^{-1}]\, z(z-1)\dots(z-k+1)\varphi(z-k)/\varphi(z). \tag{5.1}$$

See [**M**], I.7, Example 7, pages 117–118 (beware that characters are not normalized in Macdonald's book). Now we remark that

$$z\varphi(z-1)/\varphi(z) = 1/G_\lambda(z+n-1) = H_\lambda(z+n-1)$$

therefore

$$(n)_k \chi_\lambda(c_k) = -\frac{1}{k}[z^{-1}]\, H_\lambda(z+n-1)\dots H_\lambda(z+n-k)$$

Using the invariance of the residue under translation of the variable one gets

$$(n)_k \chi_\lambda(c_k) = -\frac{1}{k}[z^{-1}]\, H_\lambda(z)\dots H_\lambda(z-k+1).$$

Comparing with 2.2 we deduce the following formula for Kerov's polynomials.

Theorem 5.1. *Consider the formal power series*

$$H(z) = z - \sum_{j=2}^{\infty} B_j z^{1-j}.$$

Define

$$\Sigma_k = -\frac{1}{k}[z^{-1}]H_\lambda(z)\dots H_\lambda(z-k+1)$$

and

$$R_{k+1} = -\frac{1}{k}[z^{-1}]\, H_\lambda(z)^k$$

then the expression of Σ_k in terms of the R_k's is given by Kerov's polynomials.

This formula for computing Kerov's polynomials was shown to me by A. Okounkov [**O**]. It seems plausible that S. Kerov was aware of this (see especially the account of Kerov's central limit theorem in [**IO**]). It is much easier to implement, than the algorithm given by the proof in Section 4. We give the result of some Maple computations in Section 8.

6. Computation of some coefficients of Kerov's polynomials

For a term $R_2^{k_2}\dots R_r^{k_r}$ define its *degree* by $\sum_j k_j$ and its *weight* is $\sum_j j\,k_j$. It is clear from sign considerations that in the expansion of Σ_k only terms of weight having the opposite parity of k occur. The term of highest weight is R_{k+1} and it is the only term with this weight, as follows from Theorem 5.1.

We shall first be interested in the terms of degree one.

Theorem 6.1. *The coefficient of R_{k+1-2l} in Σ_k is equal to the number of cycles $c \in S_k$, of length k, such that $(1\,2\dots k)\,c$ has $k-2l$ cycles.*

In order to prove the theorem we shall compute the generating function for the linear coefficients, using Theorem 5.1 formula. Since we are interested only in linear terms, we see that the formula expressing Σ_k in terms of R_j is the same as the one in terms of B_j. Put $B_i = tx^{i-1}$. We shall find the coefficient of z^{-1} in $H(z)H(z-1)\dots H(z-k+1)$, keeping only the terms with degree one in t. One has $H(z) = z - tx/(z-x)$, therefore

$$[t]\,H(z)H(z-1)\dots H(z-k+1) = x\sum_{j=0}^{k-1}\frac{z(z-1)\dots(z-k+1)}{(z-k)(z-x-j)}$$

Using again invariance of residue by translation one obtains

$$\sum_l x^{2l}[R_{k+1-2l}]\,\Sigma_k = \frac{1}{k}\sum_{j=0}^{k-1}\prod_{l=0}^{k-1}(x+j-l) = \frac{1}{k}\sum_{l=0}^{k-1}Q_k(x-l) \tag{6.1}$$

where $Q_k(x) = x(x+1)\dots(x+k-1)$.

Denote by d_λ the dimension of the irreducible representation with Young diagram λ.

Lemma 6.2. *Let λ be a Young diagram with k cells, let*

$$P_\lambda(x) = \sum_{\square\in\lambda}(x + c(\square))$$

be its content polynomial, and $c(\sigma)$ = number of cycles of σ, then

$$\sum_{\sigma\in S_k} d_\lambda\chi_\lambda(\sigma)x^{l(\sigma)} = P_\lambda(x)$$

See [**M**], I.1, Example 11, I.3, Example 4, and (7.7).

Let c_k be the cycle $(1\,2\dots k)$, by the orthogonality relations for characters and Lemma 6.2, one has

$$\frac{1}{k!}\sum_\lambda d_\lambda\chi_\lambda(c_k)P_\lambda(x)P_\lambda(y) = \sum_{\sigma\in S_k} x^{l(\sigma^{-1}c_k)}y^{l(\sigma)} \tag{6.2}$$

Let us compute the coefficient of y in the left hand side of (6.2). Only hook diagrams $\lambda = (k-l, 1^l)$ for $l = 0,\dots,k-1$, contribute and for such a diagram $d_\lambda\chi_\lambda(c_k) = \binom{k-1}{l}(-1)^l$. One has $P_\lambda(x) = Q(x-l)$, the coefficient of y in $\frac{1}{k!}P_\lambda(y)$ is $\frac{1}{k}\binom{k-1}{l}^{-1}$, and $P_\lambda(x) = Q(x-l)$ therefore we find formula (6.1) for the left hand side. Comparing with the right hand side we get Theorem 6.1. □

Theorem 6.1 has also been proved by R. Stanley [**St1**] by a closely related method.

THEOREM 6.3. *The coefficient of* $R_{k-3}R_2$ *in* Σ_k *(for* $k > 5$*) is* $(k+1)k(k-1)(k-4)/12$.

Again this follows from Theorem 5.1 through some lengthy, but straightforward computations which are omitted.

More generally, based on numerical investigations, we conjecture the following formula for terms of weight $k-1$.

CONJECTURE 6.4. *The coefficient of* $R_2^{l_2}\ldots R_s^{l_s}$ *in* Σ_k, *with* $k = 2l_2 + 3l_3 + \ldots + sl_s + 1$ *is equal to*

$$\frac{(k+1)k(k-1)}{24}\frac{(l_2+\ldots+l_s)!}{l_2!\ldots l_s!}\prod_{j=2}^{s}(j-1)^{l_j}$$

The validity of this conjecture has been checked up to $k = 15$. A proof for the other cases (at least for degree two terms) can presumably be given using Theorem 5.1 but the computations become quickly very involved. No such simple product formula seems to be available for the general term.

Another natural conjecture is that all coefficients in Kerov's formula are non negative integers, which also has been checked up to $k = 15$. See the next Section for more on this.

7. Connection with the Cayley graph of symmetric group

Let us explore more thoroughly the connections between the M_k, B_k, R_k and Σ_k. Formulas (2.5) and (2.6) provide a natural combinatorial model for expressing moments in terms of free or Boolean cumulants. We will be looking for similar models for expressing the other connections. Observe first that for all measures associated with Young diagrams one has $M_1 = B_1 = R_1 = 0$. We will restrict ourselves to this case in the following.

Let us apply Kerov's formula to the trivial character. As we shall see, this will give a lot of information. The probability measure associated with the trivial character is $\frac{n}{n+1}\delta_{-1} + \frac{1}{n+1}\delta_n$ with Cauchy transform

$$G(z) = \frac{z-n+1}{(z+1)(z-n)}.$$

The corresponding moments are

$$M_k = \frac{n}{n+1}(-1)^k + \frac{1}{n+1}n^k = n\frac{n^{k-1}-(-1)^{k-1}}{n-(-1)} = (-1)^{k-1}\sum_{j=1}^{k-1}(-n)^j$$

We will find it useful to interpret this as the generating function for the rank in a totally ordered set with $k-1$ elements. We take this totally ordered set as the set I_{k-1} of partitions of $\{1,\ldots,k-1\}$ given by $\{1,2,\ldots,j\};\{j+1\};\{j+2\};\ldots\{k-1\}$ and the rank is the number of parts.

$$M_k = (-1)^{k-1}\sum_{\pi\in I_{k-1}}(-n)^{|\pi|} \tag{7.1}$$

The k^{th} free cumulant of this measure is the generating function

(7.2)

$$R_k = (-1)^{k-1}\sum_{l=1}^{k-1}\frac{1}{k-1}\binom{k-1}{l-1}\binom{k-1}{l}(-n)^l = (-1)^{k-1}\sum_{\pi\in NC(k-1)}(-n)^{|\pi|}.$$

For the Boolean cumulants one finds

(7.3)

$$B_k = n(n-1)^{k-2} = (-1)^{k-1}\sum_{j=1}^{k-1}\binom{k-2}{j-1}(-n)^j = (-1)^{k-1}\sum_{\pi\in B(k-1)}(-n)^{|\pi|}.$$

The evaluation of the trivial character on Σ_k gives $n(n-1)\dots(n-k+1)$. This is the generating function (recall that $l(\sigma)$ is the number of cyles of σ)

$$\Sigma_k = (-1)^{k-1}\sum_{\sigma\in S_k}(-n)^{l(\sigma)}$$

Let us now concentrate on the formula expressing the free cumulants in terms of Boolean cumulants, and the Boolean cumulants in terms of moments. The first such expressions are

$$\begin{array}{rcl}
R_2 &=& B_2\\
R_3 &=& B_3\\
R_4 &=& B_4 - B_2^2\\
R_5 &=& B_5 - 3B_2B_3\\
R_6 &=& B_6 - 4B_2B_4 - 2B_3^2 + 2B_2^3\\
R_7 &=& B_7 - 5B_2B_5 - 5B_3B_4 + 10B_2^2B_3
\end{array}$$

$$\begin{array}{rcl}
B_2 &=& M_2\\
B_3 &=& M_3\\
B_4 &=& M_4 - M_2^2\\
B_5 &=& M_5 - 2M_2M_3\\
B_6 &=& M_6 - 2M_2M_4 - M_3^2 + M_2^3\\
B_7 &=& M_7 - 2M_2M_5 - 2M_3M_4 + 3M_2^2M_3
\end{array}$$

These formulas contain signs but we can make all coefficients positive by an overall sign change of all variables. We are going to give a combinatorial interpretation of these coefficients. Let us replace moments, free cumulants and Boolean cumulants by the values (7.1), (7.2) and (7.3). It seems natural to try to interpret the formulas expressing free cumulants as providing a decomposition of the lattice of noncrossing partitions

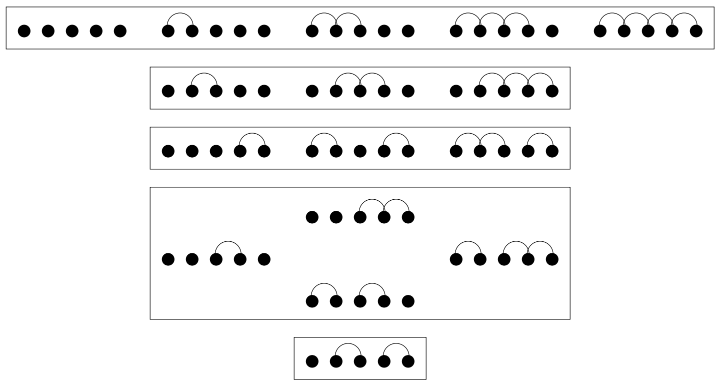
$$NC(k-1) = \cup_\pi B[\pi]$$

into a disjoint union of subsets which are products of Boolean lattices, whereas the formula expressing Boolean cumulants should come from a decomposition of the Boolean lattice

$$B(k-1) = \cup_\pi I[\pi]$$

into a union of products of totally ordered sets.

It turns out that such decompositions exist, and we shall now describe them. First we look at the formula for expressing Boolean cumulants in terms of moments. On the Boolean lattice of interval partitions, let us put a new, stronger order relation. For this new order a covers b if and only if b can be obtained from a by deleting the last element of some interval, if this interval had at least three elements, or if it is the interval $[1,2]$. The resulting decomposition of the Boolean lattice of interval partitions of $\{1,2,3,4,5\}$ is shown in the following picture. It corresponds to the formula for B_6 above. The first line in the picture corresponds to the interval M_6, the second and third line account for the two intervals M_2M_4, the fourth line for the product interval M_3^2, and the last line for the point corresponding to M_2^3. The ranking is horizontal. The proof that this decompostion yields the right interpretation of the Boolean cumulant-moment formula is easy and left to the reader.



$Fig.3$

Now let us decompose the lattice of non-crossing partitions into a union of Boolean lattices. For this we put the following new order on noncrossing partitions. A noncrossing partition a covers a noncrossing partition b if and only if b can be obtained from a by cutting a part of b between two successive elements $i, i+1$. For example we give here the list of the Boolean intervals obtained in this way in $NC(5)$, corresponding to the formula for R_6. Next to each interval we give the term to which it corresponds in the formula. Consider the Cayley graph of S_k with respect to the generating set of all transpositions. The lattice $NC(k)$ can be embedded into S_k, as the subset of elements lying on a geodesic from e to the full cycle $(12...k)$. In this embedding, the cycles of a permutation correspond to the parts of a partition, hence the functions $|\pi|$ and $l(\sigma)$. We give in the notation the cycle structure of a noncrossing partition as an element of S_k. An interval $[\sigma, \sigma']$ in which $\sigma^{-1}\sigma'$ has a cycle structure $2^{l_2}\dots k^{l_k}$ corresponds to a term $B_2^sB_3^{l_2}B_4^{l_3}\dots B_{k+1}^{l_k}$, where s is such that the rank of σ is $s+l_2+\dots+l_k$.

$[e,(1\,2\,3\,4\,5)]$	B_6
$[(1\,3),(1\,3\,4\,5)]$	B_2B_4
$[(1\,4),(1\,4\,5)(2\,3)]$	B_3^2
$[(1\,5),(1\,5)(2\,3\,4)]$	B_2B_4
$[(2\,4),(1\,2\,4\,5)]$	B_2B_4
$[(2\,5),(1\,2\,5)(3\,4)]$	B_3^2
$[(3\,5),(1\,2\,3\,5)]$	B_2B_4
$[(1\,3\,5)]$	B_2^3
$[(1\,5)(2\,4)]$	B_2^3

Observe that each interval above is in fact an interval for the Bruhat order.

Looking at the above results and at Theorem 6.1, it is tempting to try interpreting Kerov polynomials as coming from a decomposition of the symmetric group into "intervals" for some suitable order relation, in which the adjacency relation should be induced by the one of the Cayley graph. The coefficient of $R_2^{l_2}\dots R_s^{l_s}$ would count the number of intervals isomorphic to the ordered set $NC(1)^{l_2}NC(2)^{l_3}\dots NC(s-1)^{l_s}$. Such interval would be of the form $[\sigma,\sigma']$ with $|\sigma| = 1+(k+1-\sum_j l_j)/2$, and $|\sigma'| = 1+(k+1+\sum_j l_j)/2$. The obvious choice for the first term R_{k+1} in Σ_k would be to take the set of geodesics from e to $(1\,2\dots k)$. Observe however that because of sign problems we should get a "signed" covering of S_k, namely each element of S_k would be contained in a certain number of intervals and intervals corresponding to terms of even degree would give a multiplicity one while terms of odd degree would give a multiplicity -1, the sum of multiplicities would then be $+1$ for any $\sigma\in S_k$.

One way to get around this problem of signed covering would be to look at the expression of characters in terms of Boolean cumulants, where this problem disappears. One would then be lead to look for a decomposition of the Cayley graph into a union of products of Boolean lattices. It is here natural to try doing so by using the Bruhat order. Indeed some decompositions of the symmetric group into Boolean lattices have appeared in the litterature [**LS**], [**M**], but they are not the ones we are looking for, indeed by Theorem 5.1 one can compute the total number of intervals which should occur in the decomposition of S_k, and even the generating function of the number of terms according to their degrees, it is given by

$$S_{2p-1}(x) = \frac{1}{p}\binom{2p-2}{p-1}x\prod_{j=2}^{p}(x+i(i-1)) \qquad S_{2p}(x) = (2p-1)S_{2p-1}(x)$$

whereas the above decompositions have $(k-1)!$ intervals.

It has been observed by R. Stanley [**St2**], that if one evaluates the character of a cycle for a rectangular $p\times q$ Young diagram, then one can improve the formulas (7.1) to (7.3) by replacing them by their homogeneous two-variable correspondants, while the character is now given by the rhs of (6.2), with $x=-p, y=q$. This gives more evidence for the connection with the Cayley graph of symmetric group.

Let us now look at the first values of k. The cases of Σ_k for $k=2,3,4$ do not present difficulty so let us concentrate on $\Sigma_5 = R_6+15R_4+5R_2^2+8R_2$. We already have the interpretation of the terms R_6 and $8R_2$, they should correspond repectively to the interval $[e,(1\,2\,3\,4\,5)]$, of elements such that $d(e,\sigma)+d(\sigma,(1\,2\,3\,4\,5)) = 4$ (d is the distance in the Cayley graph) and the eight one point intervals $[c]$ where c is a 5-cycle whose product with $(1\,2\,3\,4\,5)^{-1}$ is a 5-cycle. It remains to cover the elements of S_5 satisfying $d(e,\sigma)+d(\sigma,(1\,2\,3\,4\,5)) = 6$ by 15 intervals isomorphic to a 3-cycle. This should yield 5 elements with $d(e,\sigma) = 3 = d(\sigma,(1\,2\,3\,4\,5))$ which are counted twice. After some guesswork the following (non unique) decomposition can be found. The 15 intervals are

$$\begin{gathered}[(1\,3)(2\,4),(1\,3\,4\,2\,5)]\\ [(1\,3\,2),(1\,3\,2\,4\,5)]\\ [(1\,4\,2),(1\,4\,2\,3\,5)]\end{gathered}$$

and their conjugates by $(1\,2\,3\,4\,5)$. The 5 elements counted twice are

$$(1\,3)(2\,5\,4)$$

which appears in the intervals $[(1\,3)(2\,4),(1\,3\,4\,2\,5)]]$ and $[(2\,5\,4),(1\,3\,2\,5\,4)]$, and all its conjugates by $(1\,2\,3\,4\,5)$.

8. Values of Σ_k for $k=7$ to 11

$$\Sigma_7 = R_8 + 70\,R_6 + 84\,R_2\,R_4 + 56\,{R_3}^2 + 14\,{R_2}^3 + 469\,R_4 + 224\,{R_2}^2 + 180\,R_2$$

$$\begin{aligned}\Sigma_8 = {} & R_9 + 126\,R_7 + 168\,R_2\,R_5 + 252\,R_3\,R_4 + 126\,{R_2}^2\,R_3 \\ & +1869\,R_5 + 2688\,R_2\,R_3 + 3044\,R_3\end{aligned}$$

$$\begin{aligned}\Sigma_9 = {} & R_{10} + 210\,R_8 + 300\,R_2\,R_6 + 480\,R_3\,R_5 + 270\,{R_4}^2 \\ & +360\,R_2\,{R_3}^2 + 270\,{R_2}^2\,R_4 + 30\,{R_2}^4 \\ & +5985\,R_6 + 10548\,R_2\,R_4 + 6714\,{R_3}^2 + 2400\,{R_2}^3 \\ & +26060\,R_4 + 14580\,{R_2}^2 + 8064\,R_2\end{aligned}$$

$$\begin{aligned}\Sigma_{10} = {} & R_{11} + 330\,R_9 + 495\,R_7\,R_2 + 825\,R_3\,R_6 + 990\,R_4\,R_5 \\ & +495\,R_5\,{R_2}^2 + 1485\,R_2\,R_3\,R_4 + 330\,{R_3}^3 + 330\,{R_2}^3\,R_3 \\ & +16401\,R_7 + 32901\,R_2\,R_5 + 46101\,R_3\,R_4 + 33000\,{R_2}^2\,R_3 \\ & +152900\,R_5 + 258060\,R_2\,R_3 + 193248\,R_3\end{aligned}$$

$$\begin{aligned}\Sigma_{11} = {} & R_{12} + 495\,R_{10} + 770\,R_8\,R_2 + 1320\,R_3\,R_7 + 1650\,R_6\,R_4 + 880\,{R_5}^2 \\ & +825\,{R_2}^2\,R_6 + 2640\,R_5\,R_2\,R_3 + 1485\,R_2\,{R_4}^2 + 1980\,{R_3}^2\,R_4 \\ & +660\,{R_2}^3\,R_4 + 1320\,{R_2}^2\,{R_3}^2 + 55\,{R_2}^5 \\ & +39963\,R_8 + 87890\,R_2\,R_6 + 130108\,R_3\,R_5 + 71214\,{R_4}^2 \\ & +105545\,{R_2}^2\,R_4 + 136345\,R_2\,{R_3}^2 + 15400\,{R_2}^4 \\ & +696905\,R_6 + 1459700\,R_2\,R_4 + 902440\,{R_3}^2 + 386980\,{R_2}^3 \\ & +2286636\,R_4 + 1401444\,{R_2}^2 + 604800\,R_2\end{aligned}$$

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