

Chapter 1

Gaussian matrix ensembles

Historically, log-potential systems first came to be studied due to an analogy with random matrices. The matrices have Gaussian elements and are Hermitian; they are then divided into three classes according to the elements being real, complex or real quaternion. The relationship of these matrix ensembles to quantum physics will be explained, and their eigenvalue probability density function (p.d.f.) calculated. A random tridiagonal matrix which interpolates continuously between the eigenvalue p.d.f.'s of the Gaussian ensembles is presented. This p.d.f. coincides with the Boltzmann factor of a certain log-potential system for general inverse temperature. In the case of Gaussian Hermitian matrices with complex elements, certain averages are related to combinatorial problems involving the embedding of maps on surfaces. Solutions are given which make use of results from the log-gas picture.

1.1 Boltzmann factors

The analogy between random matrices and log-potential systems involves the Boltzmann factor of the latter. Let us therefore begin by showing how the Boltzmann factor is computed. For this purpose, some basic theory from statistical mechanics is required.

The canonical formalism of statistical mechanics applies to any mechanical system of N particles free to move in a fixed domain Ω , in equilibrium at absolute temperature T . A fundamental postulate gives the p.d.f. for the event that the particles are at positions $\vec{r}_1, \dots, \vec{r}_N$ as

$$\frac{1}{\hat{Z}_N} e^{-\beta U(\vec{r}_1, \dots, \vec{r}_N)}.$$

Here $U(\vec{r}_1, \dots, \vec{r}_N)$ denotes the total potential energy of the system, $\beta := 1/k_B T$ (k_B is Boltzmann's constant), and the normalization \hat{Z}_N is given by

$$\hat{Z}_N = \int_{\Omega} d\vec{r}_1 \cdots \int_{\Omega} d\vec{r}_N e^{-\beta U(\vec{r}_1, \dots, \vec{r}_N)}. \quad (1.1)$$

The term $e^{-\beta U(\vec{r}_1, \dots, \vec{r}_N)}$ is referred to as the *Boltzmann factor* while $\hat{Z}_N/N! =: Z_N$ is called the (*canonical*) *partition function*.

For log-potential Coulomb systems the potential energy U is calculated according to the laws of two-dimensional electrostatics, and Ω must be one or two dimensional. The particles can be thought of as infinitely long parallel charged lines, which are perpendicular to the confining domain. In a vacuum the electrostatic potential Φ at a point $\vec{r} = (x, y)$ due to a two-dimensional unit charge at $\vec{r}' = (x', y')$ is given by the solution of the *Poisson equation*

$$\nabla_{\vec{r}}^2 \Phi(\vec{r}, \vec{r}') = -2\pi\delta(\vec{r} - \vec{r}') \quad (1.2)$$

where

$$\nabla_{\vec{r}}^2 := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

It is straightforward to verify that the solution of the Poisson equation is (see Exercises 1.1 q.1)

$$\Phi(\vec{r}, \vec{r}') = -\log(|\vec{r} - \vec{r}'|/l), \quad (1.3)$$

where l is some arbitrary length scale which will henceforth be set to unity.

A Coulomb system is said to consist of *one-component* if all N particles are of like charge, q say. To stop the particles all repelling to the boundary, a neutralizing background charge density $-q\rho_b(\vec{r})$ is imposed, with the electroneutrality condition $\int_{\Omega} \rho_b(\vec{r}) d\vec{r} = N$. The total potential energy U therefore consists of the sum of the electrostatic energy of the particle-particle interaction

$$U_1 := -q^2 \sum_{1 \leq j < k \leq N} \log|\vec{r}_k - \vec{r}_j|,$$

the particle-background interaction

$$U_2 := q^2 \sum_{j=1}^N V(\vec{r}_j) \quad \text{where} \quad V(\vec{r}_j) := \int_{\Omega} \log|\vec{r} - \vec{r}_j| \rho_b(\vec{r}) d\vec{r}, \quad (1.4)$$

and the background-background interaction

$$U_3 := -\frac{q^2}{2} \int_{\Omega} d\vec{r}' \rho_b(\vec{r}') \int_{\Omega} d\vec{r} \rho_b(\vec{r}) \log|\vec{r}' - \vec{r}| = -\frac{q^2}{2} \int_{\Omega} \rho_b(\vec{r}') V(\vec{r}') d\vec{r}'. \quad (1.5)$$

The factor of $1/2$ in U_3 is included to compensate for the double counting of the potential energy implicit in the double integration.

From this expression for U we conclude that the Boltzmann factor of a one-component log-potential Coulomb system is of the form

$$e^{-\beta U_3} \prod_{l=1}^N e^{-\Gamma V(\vec{r}_l)} \prod_{1 \leq j < k \leq N} |\vec{r}_k - \vec{r}_j|^{\Gamma} \quad (1.6)$$

where $\Gamma := q^2/k_B T$. Furthermore, for a given geometry and background density the potentials $V(\vec{r})$ and U_3 can readily be evaluated. As an illustration, we have the following result.

Proposition 1.1 *The Boltzmann factor of a one-component log-potential Coulomb system of N particles of charge $q = 1$, confined to a circle of radius R with a uniform neutralizing background, is given by*

$$R^{-N\beta/2} \prod_{1 \leq j < k \leq N} |e^{i\theta_k} - e^{i\theta_j}|^{\beta},$$

where the position of each particle has been specified in polar coordinates.

Proof It is generally true that for two points \vec{r} and \vec{r}' in the plane $|\vec{r} - \vec{r}'| = |z - z'|$ where z and z' are the corresponding points in the complex plane. Hence, if \vec{r} and \vec{r}' are both on a circle of radius R with positions specified using polar coordinates, then $|\vec{r} - \vec{r}'| = R|e^{i\theta} - e^{i\theta'}|$. Use of this formula gives the required expression for the product over pairs in (1.6). It also allows the potential $V(\vec{r})$ to be written as

$$\begin{aligned} V(\vec{r}) &= \frac{N}{2\pi R} \int_0^{2\pi} \log |Re^{i\theta'} - Re^{i\theta}| R d\theta' \\ &= N \log R + \frac{N}{2\pi} \int_0^{2\pi} \log |e^{i\theta'} - 1| d\theta'. \end{aligned}$$

But it is straightforward to show that the last integral vanishes (see Exercises 1.1 q.2), and so $V(\vec{r}) = N \log R$. Use of this result gives $U_3 = -\frac{q^2}{2} N^2 \log R$. Substituting these evaluations in (1.6) and noting that since $q = 1$, $\Gamma = \beta$ gives the desired expression for the Boltzmann factor.

The Boltzmann factor, being proportional to the p.d.f. for the location of the particles, occurs in the definition of all statistical quantities associated with the equilibrium state. In particular the *canonical average* of any function $f(\vec{r}_1, \dots, \vec{r}_N)$ is given by

$$\langle f \rangle := \frac{1}{\hat{Z}_N} \int_{\Omega} d\vec{r}_1 \dots \int_{\Omega} d\vec{r}_N f(\vec{r}_1, \dots, \vec{r}_N) e^{-\beta U(\vec{r}_1, \dots, \vec{r}_N)}. \quad (1.7)$$

1.1.1 Free energy

In the canonical formalism of statistical mechanics, the total dimensionless free energy βF is given by

$$\beta F = -\log \frac{1}{N!} \hat{Z}_N \quad (1.8)$$

and the corresponding dimensionless free energy per particle βf is given by

$$\beta f = \lim_{\substack{N, |\Omega| \rightarrow \infty \\ N/|\Omega| = \rho}} \frac{1}{N} \beta F. \quad (1.9)$$

The limit $N, |\Omega| \rightarrow \infty$, $N/|\Omega| = \rho$ (fixed) is referred to as the *thermodynamic limit*. From βf other thermodynamic quantities of interest can be computed by differentiation. In particular

$$\frac{\partial \beta f}{\partial \beta} = \lim_{\substack{N, |\Omega| \rightarrow \infty \\ N/|\Omega| = \rho}} \left(\frac{1}{N} \langle U \rangle \right) =: u \quad (1.10)$$

gives the mean energy per particle. A further differentiation gives the *specific heat* C_V at constant volume (also referred to as the heat capacity),

$$C_V/k_B = -\frac{1}{\beta^2} \frac{\partial u}{\partial \beta}. \quad (1.11)$$

We see from (1.7) and (1.10) that in terms of averages

$$C_V/k_B = \frac{1}{\beta^2} \lim_{\substack{N, \Omega \rightarrow \infty \\ N/|\Omega| = \rho}} \frac{1}{N} \left(\langle U^2 \rangle - \langle U \rangle^2 \right), \quad (1.12)$$

so in particular C_V is non-negative.

Also, in thermodynamics, the pressure is related to the free energy by $dF = -Pd|\Omega| - SdT$ (S denotes the entropy) so one has

$$P = -\left(\frac{\partial F}{\partial |\Omega|}\right)_T \sim \rho^2 \frac{\partial f}{\partial \rho}, \quad (1.13)$$

where the final expression holds in the thermodynamic limit.

1.1.2 Distribution functions

With $f = \sum_{j \neq k=1}^N \delta(\vec{r}' - \vec{r}_j) \delta(\vec{r} - \vec{r}_k)$ the canonical average (1.7) is called the two-particle distribution function $\rho_{(2)}(\vec{r}, \vec{r}')$,

$$\begin{aligned} \rho_{(2)}(\vec{r}, \vec{r}') &:= \frac{1}{\hat{Z}_N} \left\langle \sum_{\substack{j,k=1 \\ j \neq k}}^N \delta(\vec{r} - \vec{r}_j) \delta(\vec{r}' - \vec{r}_k) \right\rangle \\ &= \frac{N(N-1)}{\hat{Z}_N} \int_{\Omega} d\vec{r}_3 \cdots \int_{\Omega} d\vec{r}_N e^{-\beta U(\vec{r}, \vec{r}', \vec{r}_3, \dots, \vec{r}_N)} \end{aligned} \quad (1.14)$$

where the last equality is valid for a system of identical particles. With this assumption, the n -particle distribution is given by

$$\rho_{(n)}(\vec{r}_1, \dots, \vec{r}_n) = \frac{N(N-1) \cdots (N-n+1)}{\hat{Z}_N} \int_{\Omega} d\vec{r}_{n+1} \cdots \int_{\Omega} d\vec{r}_N e^{-\beta U(\vec{r}_1, \dots, \vec{r}_N)}. \quad (1.15)$$

We remark that $\rho_{(n)}(\vec{r}_1, \dots, \vec{r}_n) / \rho_{(n-1)}(\vec{r}_1, \dots, \vec{r}_{n-1})$ can be interpreted as the density at point \vec{r}_n given that there are particles at points $\vec{r}_1, \dots, \vec{r}_{n-1}$.

The two-particle distribution is related to thermodynamic quantities. In particular, when U is given by the sum of pair potentials $\phi(\vec{r})$, and the system is in a fluid state so that

$$\rho_{(2)}(\vec{r}, \vec{r}') = \rho_{(2)}(\vec{r} - \vec{r}', 0) =: \rho_{(2)}(\vec{r} - \vec{r}'),$$

we see from the definitions that for large $|\Omega|$

$$\langle U \rangle \sim \frac{|\Omega|}{2} \int_{\Omega} \phi(\vec{r}) \rho_{(2)}(\vec{r}) d\vec{r}. \quad (1.16)$$

But for one component charged systems, the potential energy also contains a contribution from the particle-background and background-background interactions. The above formula then requires modification: the two-particle distribution $\rho_{(2)}(\vec{r}, \vec{r}')$ is to be replaced by the truncated distribution

$$\rho_{(2)}^T(\vec{r}, \vec{r}') := \rho_{(2)}(\vec{r}, \vec{r}') - \rho_{(1)}(\vec{r}) \rho_{(1)}(\vec{r}'). \quad (1.17)$$

(see Exercises 1.1 q.5).

The distribution function $\rho_{(n)}$ does not decay for large separation between particles. However, by adding and subtracting appropriate combinations of $\rho_{(1)}, \dots, \rho_{(n-1)}$ to $\rho_{(n)}$ we can obtain a quantity, denoted $\rho_{(n)}^T$ and called the n -particle correlation function or the (fully) truncated n -particle distribution function, which will decay when two or more particles are at large separation.

The truncated two-particle distribution is given by (1.17), while the truncated three-particle distribution is given by

$$\begin{aligned} \rho_{(3)}^T(\vec{r}_1, \vec{r}_2, \vec{r}_3) &:= \rho_{(3)}(\vec{r}_1, \vec{r}_2, \vec{r}_3) - \rho_{(1)}(\vec{r}_1)\rho_{(2)}(\vec{r}_2, \vec{r}_3) - \rho_{(1)}(\vec{r}_2)\rho_{(2)}(\vec{r}_1, \vec{r}_3) \\ &\quad - \rho_{(1)}(\vec{r}_3)\rho_{(2)}(\vec{r}_1, \vec{r}_2) + 2\rho_{(1)}(\vec{r}_1)\rho_{(1)}(\vec{r}_2)\rho_{(1)}(\vec{r}_3) \end{aligned} \quad (1.18)$$

and in general

$$\rho_{(n)}^T(\vec{r}_1, \dots, \vec{r}_n) := \sum_{m=1}^n \sum_G (-1)^{m-1} (m-1)! \prod_{j=1}^m \rho_{(|G_j|)}(\vec{r}_{g_j(1)}, \dots, \vec{r}_{g_j(|G_j|)}) \quad (1.19)$$

where the sum over G is over all subdivisions of $\{1, 2, \dots, n\}$ into m subsets G_1, \dots, G_m with $G_j = \{g_j(1), \dots, g_j(|G_j|)\}$. For example, when $n = 3$ the $m = 1$ term corresponds to $G_1 = \{1, 2, 3\}$, the $m = 2$ term to $G_1 = \{1\}$, $G_2 = \{2, 3\}$ or $G_1 = \{2\}$, $G_2 = \{1, 3\}$ or $G_1 = \{3\}$, $\{1, 2\}$, and the $m = 3$ term to $G_1 = \{1\}$, $G_2 = \{2\}$, $G_3 = \{3\}$ (rearrangements of the G_j are not considered distinct). The inverse of this formula is given in Exercises 1.1 q.6.

Exercises 1.1

1.(i) By explicit differentiation show that $\Phi(\vec{r}, \vec{r}') = -\log(|\vec{r} - \vec{r}'|/l)$, satisfies the two dimensional Laplace's equation $\nabla_{\vec{r}}^2 \Phi(\vec{r}, \vec{r}') = 0$ for $\vec{r} \neq \vec{r}'$.

(ii) Use the divergence theorem in the plane

$$\int_{\mathcal{D}} \nabla^2 V(\vec{r}) d\vec{r} = \int_{\mathcal{C}} \vec{n} \cdot \nabla V(\vec{r}) d\vec{r}$$

with $V(\vec{r}) = \Phi(\vec{r}, \vec{r}')$, \mathcal{D} a disk centred on \vec{r}' and \mathcal{C} the circle which is the boundary of the disk, to conclude

$$\int_{\mathcal{D}} \nabla_{\vec{r}}^2 \Phi(\vec{r}, \vec{r}') d\vec{r} = -2\pi.$$

Relate this result to the Poisson equation (1.2).

2. Use the power series expansion of $\log(1 - z)$ for $|z| < 1$ to show that for all $|\mu| > 1$,

$$\int_0^{2\pi} \log|1 - \mu e^{i\theta}| d\theta = 0.$$

Show that this integral is equal to $2\pi \log|\mu|$ for $|\mu| > 1$ by using the result for $|\mu| < 1$, and use the continuity of the integral as a function of μ to deduce its value for $|\mu| = 1$.

3. Consider the two-dimensional one-component plasma confined to a disk. Suppose there are N mobile particles of charge q and the disk is filled with a uniform neutralizing background $\rho_b = N/\pi R^2$.

(i) With the position of the particles specified in polar coordinates, use the integral evaluations of q.2 and the definition of $V(r)$ (1.4) to show

$$V(r) = \pi\rho_b(r^2/2 + R^2 \log R - R^2/2).$$

Write down the Poisson equation satisfied by $V(r)$.

(ii) Use this expression for $V(r)$ to calculate U_3 and thus show that the Boltzmann factor is equal to

$$e^{-\Gamma N^2((1/2) \log R - 3/8)} e^{-\pi\Gamma\rho_b \sum_{j=1}^N |\vec{r}_j|^2/2} \prod_{1 \leq j < k \leq N} |\vec{r}_k - \vec{r}_j|^\Gamma. \quad (1.20)$$

4. Show that the total electrostatic energy $U_1 + U_2 + U_3$ of a one-component system with pair potential $\Phi(\vec{r}, \vec{r}') = -\log|\vec{r} - \vec{r}'|$ is related to the total electrostatic energy $U_1(l) + U_2(l) + U_3(l)$ of a one-component system with pair potential $\Phi(\vec{r}, \vec{r}') = -\log(|\vec{r} - \vec{r}'|/l)$ by

$$U_1(l) + U_2(l) + U_3(l) = U_1 + U_2 + U_3 - \frac{q^2 N}{2} \log l.$$

5.(i) For the one-component plasma with ρ_b constant show that to leading order in the volume $|\Omega|$

$$q^2 \int_{\Omega} V(\vec{r}) d\vec{r} \sim -|\Omega| \rho_b \int_{\Omega} \phi(\vec{r}) d\vec{r},$$

where $\phi(r) = -q^2 \log r$.

(ii) Use (i) to show that for ρ_b constant, to leading order in $|\Omega|$ the particle-background potential U_2 and background-background potential U_3 are such that

$$\langle U_2 + U_3 \rangle \sim -\frac{1}{2} \rho_b^2 |\Omega| \int_{\Omega} \phi(\vec{r}) d\vec{r},$$

and thus deduce that for the one-component plasma $\rho_{(2)}(\vec{r})$ needs to be replaced by $\rho_{(2)}^T(\vec{r})$ in (1.16).

6. Define

$$u_n[a] = \int_{\Omega} d\vec{r}_1 \cdots \int_{\Omega} d\vec{r}_n \prod_{l=1}^n a(\vec{r}_l) \rho_{(n)}(\vec{r}_1, \dots, \vec{r}_n), \quad v_n[a] = \int_{\Omega} d\vec{r}_1 \cdots \int_{\Omega} d\vec{r}_n \prod_{l=1}^n a(\vec{r}_l) \rho_{(n)}^T(\vec{r}_1, \dots, \vec{r}_n)$$

and introduce the generating functions

$$U[z; a] = 1 + \sum_{n=1}^{\infty} \frac{u_n[a]}{n!} z^n, \quad V[z; a] = \sum_{n=1}^{\infty} \frac{v_n[a]}{n!} z^n.$$

(i) Show that

$$U[z; a] = \frac{1}{\hat{Z}_N} \int_{\Omega} d\vec{r}_1 \cdots \int_{\Omega} d\vec{r}_N \prod_{l=1}^N (1 + za(\vec{r}_l)) e^{-\beta U(\vec{r}_1, \dots, \vec{r}_N)}.$$

(ii) Note from (1.19) that

$$v_n[a] = n! \sum_{m=1}^n \frac{(-1)^{m-1}}{m} \sum_{\substack{k_1, \dots, k_m \geq 1 \\ k_1 + \dots + k_m = n}} \prod_{j=1}^m \frac{u_{k_j}[a]}{k_j!}.$$

(iii) Use the result of (ii) to show

$$V[z; a] = \log U[z; a],$$

and from this deduce that

$$u_n[a] = n! \sum_{m=1}^n \sum_{\substack{k_1, \dots, k_m \geq 1 \\ k_1 + \dots + k_m = n}} \frac{1}{m!} \frac{v_{k_1}[a] \cdots v_{k_m}[a]}{k_1! \cdots k_m!},$$

or equivalently

$$\rho_{(n)}(x_1, \dots, x_n) = \sum_{m=1}^n \sum_G \prod_{j=1}^m \rho_{(|G_j|)}^T(x_{g_j(1)}, \dots, x_{g_j(|G_j|)}).$$

1.2 Random real symmetric matrices

At the beginning of this chapter it was commented that the study of log-potential systems was initiated by an analogy with the eigenvalue p.d.f. of some random matrices. In this and the following two sections we will introduce three types of Gaussian random matrix ensembles, explain their relevance to quantum physics and calculate the corresponding eigenvalue p.d.f.'s. The analogy with the Boltzmann factor of a log-potential Coulomb system will be discussed in Section 1.5.

The first random matrix ensemble to be considered is specified as follows.

Definition 1.1 *A random real symmetric $N \times N$ matrix \mathbf{X} is said to belong to the Gaussian orthogonal ensemble (GOE) if the diagonal and upper diagonal elements are independently chosen with p.d.f.'s*

$$\frac{1}{\sqrt{2\pi}}e^{-x_{jj}^2/2} \quad \text{and} \quad \frac{1}{\sqrt{\pi}}e^{-x_{jk}^2}$$

respectively.

The p.d.f.'s of Definition 1.1 are examples of the normal (or Gaussian) distribution

$$\frac{1}{\sqrt{2\pi\sigma^2}}e^{-(x-\mu)^2/2\sigma^2}$$

which is to be denoted $N[\mu, \sigma]$.

The joint p.d.f. of all the independent elements

$$\begin{aligned} P(\mathbf{X}) &:= \prod_{j=1}^N \frac{1}{\sqrt{2\pi}}e^{-x_{jj}^2/2} \prod_{1 \leq j < k \leq N} \frac{1}{\sqrt{\pi}}e^{-x_{jk}^2} = A_N \prod_{j,k=1}^N e^{-x_{jk}^2/2} \\ &= A_N e^{-\sum_{j,k=1}^N x_{jk}^2/2} = A_N e^{-(1/2)\text{Tr}\mathbf{X}^2}, \end{aligned} \quad (1.21)$$

where A_N is the normalization and Tr denotes the trace, has some noteworthy features which are of relevance to the application of the GOE to quantum physics. These features are given by the following results (Porter [175]).

Proposition 1.2 *Let $\mathbf{R}(\mathbf{X})$ be any real orthogonal (symmetric) $N \times N$ matrix. Then $P(\mathbf{R}^T \mathbf{X} \mathbf{R}) = P(\mathbf{X})$. Furthermore, the most general p.d.f. satisfying this equation which has the factorization property $P(\mathbf{X}) = \prod_{1 \leq j < k \leq N} f(x_{jk})$ for f differentiable is*

$$P(\mathbf{X}) = A e^{-a \sum_{j,k=1}^N (x_{jk})^2 - b \sum_{j=1}^N x_{jj}} = A e^{-a \text{Tr}(\mathbf{X}^2) - b \text{Tr}\mathbf{X}}.$$

Proof See Exercises 1.2 q.1.

Proposition 1.3 *Define the entropy S of the joint p.d.f. P of the independent elements of \mathbf{X} by $S[P] := -\int P \log P \mu(d\mathbf{X}) =: -\langle \log P \rangle_P$ where $\mu(d\mathbf{X}) := \prod_{1 \leq j < k \leq N} dx_{jk}$. Then P as given by (1.21) maximizes S subject to the constraint $\langle \text{Tr}\mathbf{X}^2 \rangle_P = N^2$.*

Proof Because of the constraint on the second moment, and the normalization constraint, we can write

$$S[P] = -\langle \log P \rangle_P - \lambda \left(\langle \text{Tr} \mathbf{X}^2 \rangle_P - N^2 \right) + (\log A + 1) \left(\langle 1 \rangle_P - 1 \right)$$

where λ and $-(\log A + 1)$ are Lagrange multipliers. The condition for a maximum is $\delta S = 0$, where the variation is made with respect to P . This gives

$$-\log P - \lambda \text{Tr} \mathbf{X}^2 + \log A = 0$$

and thus $P = Ae^{-\lambda \text{Tr} \mathbf{X}^2}$. The value of λ is determined to be $1/2$ from the given constraint.

From these properties an understanding of the applicability of the GOE in the study of quantum energy spectra can be obtained. However as a further prerequisite some theory from quantum mechanics is required.

1.2.1 Time reversal in quantum systems

First it is necessary to understand the relevance of an $N \times N$ matrix to quantum energy spectra. A basic axiom of quantum mechanics says the energy spectrum of a quantum system is given by the eigenvalues of its (Hermitian) Hamiltonian operator H . Now, to model the discrete portion of the spectrum of a complicated quantum system, a reasonable approximation is to replace the in general infinite dimensional operator H by a finite dimensional $N \times N$ Hermitian matrix, which has a discrete spectrum only.

Next we need to understand the significance of real symmetric matrices in quantum mechanics. In general the structure of a matrix modelling H is constrained by the symmetries of H .

Definition 1.2 A quantum Hamiltonian H is said to have a symmetry A if

$$[H, A] = 0,$$

where $[\cdot, \cdot]$ denotes the commutator.

One basic symmetry of most quantum systems is time reversal.

Definition 1.3 A general time reversal operator T is any antiunitary operator i.e. a unitary operator which has the property

$$T \left(a|\psi_1\rangle + b|\psi_2\rangle \right) = \bar{a}T|\psi_1\rangle + \bar{b}T|\psi_2\rangle,$$

where a and b are scalars and $\bar{\alpha}$ denotes the complex conjugate of α .

Hence we say a quantum system has a time reversal symmetry if the Hamiltonian commutes with an antiunitary operator.

Study of time reversal operators in the context of physical systems (see e.g. Messiah [154, pg. 671], Haake [103, chapter 2]) further restricts their form. For systems with an even number or no spin $1/2$ particles, it is required that

$$T^2 = 1,$$

while for a finite dimensional system with an odd number of spin 1/2 particles

$$T^2 = -1 \quad \text{and} \quad T = \mathbf{Z}_{2N}K$$

where \mathbf{Z}_{2N} is a $2N \times 2N$ block diagonal matrix with each 2×2 diagonal block given by

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad (1.22)$$

(a tensor product formula for \mathbf{Z}_{2N} is given in Exercises 1.2 q.2) and K is the complex conjugation operator. Real symmetric matrices arise in the former situation.

Proposition 1.4 *Let H be a quantum Hamiltonian which is invariant with respect to a time reversal symmetry T , where T has the additional property $T^2 = 1$. Then H can always be given a T -invariant orthogonal basis, and with respect to this basis the (in general infinite) matrix representation of H is real.*

Proof See Exercises 1.2 q.3.

The above result tells us that a matrix chosen to model the discrete energy spectra of a quantum system with a time reversal symmetry T such that $T^2 = 1$ must be real symmetric. A further general property in quantum mechanics is that two operators related by a similarity transformation of unitary operators are equally valid descriptions of the operator, in that all observables are the same for both operators. A requirement of (1.21) is therefore that any two real symmetric matrices related by a similarity transformation of unitary matrices must have the same p.d.f. for the elements. Now, for the two real symmetric matrices to be so related the unitary matrix must be real orthogonal (or i times a real orthogonal matrix; see Exercises 1.2 q.4). Thus this requirement is guaranteed by Proposition 1.2.

We are assuming no information on the Hamiltonian other than the time reversal symmetry. Proposition 1.3 says that the p.d.f. (1.21) is the most random subject to the given constraint, in that it maximizes the entropy.

These considerations thus show the applicability of the GOE in the study of quantum spectra. Explicitly, it is hypothesized that the statistical properties of the highly excited states of a complex quantum system with a time reversal symmetry $T^2 = 1$ coincide with the statistical properties of the bulk eigenvalues from large GOE matrices. Here it is assumed that both spectra have been scaled (technically referred to as *unfolded*) so that the mean spacing is unity. The meaning of a complex quantum system requires further explanation. Wigner first made this hypothesis for the spectra of heavy nuclei in the 1950's. In 1984 Bohigas, Giannoni and Schmidt made the same hypothesis for a single particle quantum billiard system, provided the underlying classical mechanics is chaotic and the system has a time reversal symmetry. It is of interest to note that a GOE hypothesis also applies to eigenmodes of microwave cavities (this is not surprising as the Helmholtz equation is formally equivalent to the stationary Schrödinger equation), and also to the eigenmodes of systems governed by classical wave equations — vibrations of irregular shaped metal plates, electromechanical eigenmodes of aluminium and quartz blocks, amongst other examples. (For references to the original literature, and an extended discussion of GOE hypotheses, see [100].)

Exercises 1.2

1. The objective of this exercise is to prove Proposition 1.2.

(i) Note that the invariance $P(\mathbf{R}^T \mathbf{X} \mathbf{R}) = P(\mathbf{X})$ requires that P be a symmetric function of the eigenvalues, and thus a function of $\text{Tr}(\mathbf{X}^k)$ $k = 1, 2, \dots$. Because these traces are symmetric in the diagonal elements of the matrix, and symmetric in the off diagonal elements, conclude that we must have $f_{jj} = f$, $f_{jk} = g$ ($j < k$) for some f and g .

(ii) Choose

$$\mathbf{R} = \begin{bmatrix} 1 & \epsilon & 0 & \dots & 0 \\ -\epsilon & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

where $|\epsilon| \ll 1$. Ignoring terms $O(\epsilon^2)$, show that

$$\mathbf{R}^{-1} \mathbf{X} \mathbf{R} = \begin{bmatrix} x_{11} - 2\epsilon x_{12} & x_{12} + \epsilon(x_{11} - x_{22}) & x_{13} - \epsilon x_{23} & \dots & x_{1N} - \epsilon x_{2N} \\ * & x_{22} + 2\epsilon x_{12} & x_{23} + \epsilon x_{13} & \dots & x_{2N} + \epsilon x_{1N} \\ * & * & x_{33} & \dots & x_{3N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \dots & x_{NN} \end{bmatrix}$$

where the elements $*$ are such that the matrix is symmetric.

(iii) Use the result of (ii) to show that at first order in ϵ the requirement

$$\prod_{j=1}^N f(x_{jj}) \prod_{1 \leq j < k \leq N} g(x_{jk}) = \prod_{j=1}^N f(\tilde{x}_{jj}) \prod_{1 \leq j < k \leq N} g(\tilde{x}_{jk}),$$

where $\tilde{x}_{jk} := [\mathbf{R}^{-1} \mathbf{X} \mathbf{R}]_{jk}$ implies

$$\frac{(x_{11} - x_{22})g'(x_{12})}{g(x_{12})} - 2 \frac{x_{12}f'(x_{11})}{f(x_{11})} + 2 \frac{x_{12}f'(x_{22})}{f(x_{22})} - \sum_{j=3}^N \left(\frac{x_{2j}g'(x_{1j})}{g(x_{1j})} - \frac{x_{1j}g'(x_{2j})}{g(x_{2j})} \right) = 0,$$

which in turn, by separation of variables, implies

$$\frac{x_{2j}g'(x_{1j})}{g(x_{1j})} - \frac{x_{1j}g'(x_{2j})}{g(x_{2j})} = \alpha, \quad j \geq 3$$

for some constant α .

(iv) By choosing $x_{1j} = x_{2j}$ in the last equation, specify α and further separate variables to conclude

$$\frac{g'(x_{1j})}{g(x_{1j})x_{1j}} = -b$$

for some constant b .

(v) Solve the above differential equation and thus, after making use of the first sentence in (i), deduce the result of the proposition.

2. Let $\mathbf{A} = [a_{ij}]$ be a $p \times q$ matrix and $\mathbf{B} = [b_{i'j'}]$ be a $r \times s$ matrix. The tensor product, denoted $\mathbf{A} \otimes \mathbf{B}$ is the $pr \times qs$ matrix with elements

$$(\mathbf{A} \otimes \mathbf{B})_{ii',jj'} = a_{i,j}b_{i',j'}$$

and thus

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & \cdots & a_{1q}\mathbf{B} \\ \vdots & \vdots & \ddots & \vdots \\ a_{p1}\mathbf{B} & a_{p2}\mathbf{B} & \cdots & a_{pq}\mathbf{B} \end{bmatrix}.$$

With \mathbf{Z}_{2N} defined as above (1.22), show that

$$\mathbf{Z}_{2N} = \mathbf{1}_N \otimes \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \quad (1.23)$$

3. Let

$$\psi_1 = \alpha_1 \phi_1 + T(\alpha_1 \phi_1)$$

where α_1 is a scalar, ψ_1 and ϕ_1 are vectors, T is antiunitary and $T^2 = 1$. Note that $T\psi_1 = \psi_1$. Here Proposition 1.4 will be established.

(i) From the antiunitarity property it follows that in general $\langle u|Tv \rangle = \overline{\langle Tu|v \rangle}$. Use this to show that $\langle u|v \rangle = \overline{\langle Tu|Tv \rangle}$.

(ii) Suppose ϕ_2 is orthogonal to ψ_1 . Use (i) to show that $\psi_2 := \alpha_2 \phi_2 + T(\alpha_2 \phi_2)$ is orthogonal to ψ_1 , and note how this construction can be used to create an orthogonal basis of vectors with the T -invariance property $T\psi_n = \psi_n$.

(iii) Consider a Hamiltonian H which has the symmetry T . Use the above properties of T to show that with respect to the basis $\{\psi_n\}$ the matrix elements $\langle \psi_m | H \psi_n \rangle$ are real.

4. Let \mathbf{X} be an arbitrary real symmetric $N \times N$ matrix and suppose

$$\mathbf{X}' = \mathbf{U}^{-1} \mathbf{X} \mathbf{U}$$

where \mathbf{U} is unitary and \mathbf{X}' is real symmetric. Assume that the only symmetry of \mathbf{X} and \mathbf{X}' in general (other than some constant times the identity) is the time reversal operator T with $T^2 = 1$.

(i) Deduce that $T\mathbf{U}T^{-1}\mathbf{U}^{-1}$ commutes with \mathbf{X} .

(ii) Use (i) to show $T\mathbf{U} = c\mathbf{U}T$ and take the inverse of this equation to conclude $c = \pm 1$.

(iii) Use (ii) and q.3(i) to show that with respect to the T invariant basis $\{\psi_n\}$

$$\langle \psi_n | \mathbf{U} \psi_m \rangle = \overline{c \langle \psi_n | \mathbf{U} \psi_m \rangle}.$$

Hence conclude that \mathbf{U} has either real elements ($c = 1$) or pure imaginary elements ($c = -1$) and is thus either a real orthogonal matrix or i times a real orthogonal matrix.

1.3 The eigenvalue p.d.f. for the GOE

The p.d.f. for the elements of the matrices in the GOE is given by (1.21). We want to calculate the corresponding eigenvalue p.d.f. This was first accomplished as long ago as 1939 [109]. We will follow a more recent treatment (Muirhead [157]).

The new variables and the final expression

The p.d.f. (1.21) has $N(N+1)/2$ independent variables, whereas there are only N eigenvalues, $\lambda_1 \leq \cdots \leq \lambda_N$ say. The remaining variables are linear combinations of the independent elements

of the eigenvectors, denoted $p_1, \dots, p_{N(N-1)/2}$ say. Our task is to change variables

$$\exp\left(-\frac{1}{2}\text{Tr}(\mathbf{X}^2)\right) \prod_{1 \leq j < k \leq N} dx_{jk} = \exp\left(-\frac{1}{2} \sum_{l=1}^N \lambda_l^2\right) |J| \prod_{j=1}^N d\lambda_j \prod_{j=1}^{N(N-1)/2} dp_j$$

where the Jacobian is given by

$$J := \det \begin{bmatrix} \frac{\partial x_{11}}{\partial \lambda_1} & \frac{\partial x_{12}}{\partial \lambda_1} & \cdots & \frac{\partial x_{NN}}{\partial \lambda_1} \\ \frac{\partial x_{11}}{\partial \lambda_2} & \frac{\partial x_{12}}{\partial \lambda_2} & \cdots & \frac{\partial x_{NN}}{\partial \lambda_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_{11}}{\partial p_{N(N-1)/2}} & \frac{\partial x_{12}}{\partial p_{N(N-1)/2}} & \cdots & \frac{\partial x_{NN}}{\partial p_{N(N-1)/2}} \end{bmatrix}.$$

Thus we must evaluate the Jacobian and then integrate over the variables $p_1, \dots, p_{N(N-1)/2}$ to obtain the eigenvalue p.d.f.

Below we will show that J factorizes,

$$J = \prod_{1 \leq j < k \leq N} (\lambda_k - \lambda_j) f(p_1, \dots, p_{N(N-1)/2})$$

so the integration over the variables $p_1, \dots, p_{N(N-1)/2}$ only alters the normalization constant. Hence the final expression for the eigenvalue p.d.f. of the GOE is

$$\frac{1}{C_N} e^{-\frac{1}{2} \sum_{j=1}^N \lambda_j^2} \prod_{1 \leq j < k \leq N} |\lambda_k - \lambda_j| \quad (1.24)$$

where C_N is the normalization constant.

From the viewpoint of application to quantum mechanics, the important feature is the product of differences due to the Jacobian. In Chapter 5 we will see that the n -point correlations are determined entirely by the product of differences, in the sense the same so called bulk correlations result if the one body terms $e^{-\lambda^2/2}$ are replaced by some different functional forms $e^{-V(\lambda)/2}$, provided the local density is constant. This gives rise to the notion [24] that the essential feature of a random matrix hypothesis applying to a quantum system is that the spectral correlations are geometrical, meaning that they are due to this Jacobian.

1.3.1 Wedge products

In the theory of multivariable calculus the wedge product operation is defined to give to signed volume element in the tangent space at a point in the manifold.

Definition 1.4 *Let p be a point on a manifold with coordinates u_1, \dots, u_N , and let P be the orientated parallelepiped in the tangent space determined by the vectors $\vec{r}_1, \dots, \vec{r}_N$. Then*

$$du_1 \wedge \cdots \wedge du_N =: \bigwedge_{j=1}^N du_j := \det[du_i(\vec{r}_j)]_{i,j=1,\dots,N} \quad (1.25)$$

where it is understood that all quantities are with reference to p and P .

Note that it follows from (1.25) that

$$\int_{\Omega} f(u_1, \dots, u_N) du_1 \wedge \dots \wedge du_N = \int_{\Omega} f(u_1, \dots, u_N) du_1 \cdots du_N.$$

When changing variables from $\{u_1, \dots, u_N\}$ to $\{v_1, \dots, v_N\}$ the fundamental formula

$$du_i = \sum_{l=1}^N \frac{\partial u_i}{\partial v_l} dv_l$$

applies. Substituting this in (1.25), and noting the factorization

$$\left[\sum_{l=1}^N \frac{\partial u_i}{\partial v_l} dv_l(\vec{r}_j) \right]_{i,j=1,\dots,N} = \left[\frac{\partial u_i}{\partial v_j} \right]_{i,j=1,\dots,N} \left[dv_i(\vec{r}_j) \right]_{i,j=1,\dots,N}$$

shows

$$\bigwedge_{j=1}^N du_j = \det \left[\frac{\partial u_i}{\partial v_j} \right]_{i,j=1,\dots,N} \bigwedge_{j=1}^N dv_j. \quad (1.26)$$

The determinant in (1.26) is precisely the Jacobian for the change of variables. The practical use of calculating Jacobians from this formula relies on an alternative way of calculating the l.h.s. of (1.26) in terms of $\{v_j\}$. For the problem at hand, this in turn is done by using the special feature that all the variables are connected by matrix relations. The following definitions are useful.

Definition 1.5 For any $N \times N$ matrix $\mathbf{X} = [x_{jk}]$, the matrix of differentials is defined as

$$d\mathbf{X} = \begin{bmatrix} dx_{11} & dx_{12} & \dots & dx_{1N} \\ dx_{21} & dx_{22} & \dots & dx_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ dx_{N1} & dx_{N2} & \dots & dx_{NN} \end{bmatrix}.$$

With this definition the usual product rule for differentiation holds,

$$d(\mathbf{X}\mathbf{Y}) = d\mathbf{X}\mathbf{Y} + \mathbf{X}d\mathbf{Y}.$$

Definition 1.6 The symbol $(d\mathbf{X})$ denotes the wedge product of the independent elements of $d\mathbf{X}$. In particular, if \mathbf{X} is a symmetric matrix,

$$(d\mathbf{X}) = \bigwedge_{1 \leq j < k \leq N} dx_{jk},$$

while if $\mathbf{X} = [x_{jk} + iy_{jk}]_{j,k=1,\dots,N}$ is Hermitian ($x_{jk} = x_{kj}$, $y_{kj} = -y_{jk}$),

$$(d\mathbf{X}) = \bigwedge_{j=1}^N dx_{jj} \bigwedge_{1 \leq j < k \leq N} dx_{jk} dy_{jk}.$$

In integration formulas only the absolute value of the Jacobian occurring in the change of variables formula (1.26) is required, so consequently there is no need to strictly adhere to the ordering of wedge products specified in Definition 1.6 (according to the definition, reversing the order of two differentials changes the sign of the wedge product). Because of this, any overall factor of -1 will be ignored in subsequent formulas involving $(d\mathbf{X})$. With this convention $(d\mathbf{X})$ will be referred to as a *volume form*, or *volume measure*.

In preparation for the calculation of J , we note a result for the wedge product $(\mathbf{A}^T d\mathbf{M} \mathbf{A})$ where \mathbf{A} is a real $N \times N$ matrix and M is a real symmetric $N \times N$ matrix [157].

Proposition 1.5 *Let \mathbf{A} and \mathbf{M} be real $N \times N$ matrices, and suppose furthermore that M is symmetric. We have*

$$(\mathbf{A}^T d\mathbf{M} \mathbf{A}) = (\det \mathbf{A})^{N+1} (d\mathbf{M}).$$

Proof We note from Definition 1.6 that

$$(\mathbf{A}^T d\mathbf{M} \mathbf{A}) = p(\mathbf{A})(d\mathbf{M}), \quad (1.27)$$

where p is a polynomial in the elements of \mathbf{A} . Furthermore, if \mathbf{B} is also an $N \times N$ matrix, then

$$(\mathbf{B}^T \mathbf{A}^T d\mathbf{M} \mathbf{A} \mathbf{B}) = p(\mathbf{B})(\mathbf{A}^T d\mathbf{M} \mathbf{A}) = p(\mathbf{B})p(\mathbf{A})(d\mathbf{M})$$

so we must have

$$p(\mathbf{A}\mathbf{B}) = p(\mathbf{A})p(\mathbf{B}),$$

for arbitrary \mathbf{A} and \mathbf{B} . But it is known [141] that the only polynomial in the matrix elements satisfying such a factorization is

$$p(\mathbf{A}) = (\det \mathbf{A})^k, \quad k \in \mathbb{Z}_{\geq 0}.$$

The value of k can be determined by making the special choice $\mathbf{A} = \text{diag}(a, 1, \dots, 1)$ in (1.27).

For an alternative proof of this result, see Exercises 1.4 q.2.

1.3.2 Calculation of the Jacobian

From Definition 1.6 and (1.26) we see

$$J \prod_{i=1}^N d\lambda_i \prod_{j=1}^{N(N-1)/2} dp_j = (d\mathbf{X}).$$

To calculate $(d\mathbf{X})$ in terms of the eigenvalues and eigenvectors we use the fact that all symmetric matrices are orthogonally diagonalizable to write

$$\mathbf{X} = \mathbf{R}\mathbf{L}\mathbf{R}^T \quad (1.28)$$

where \mathbf{L} is a diagonal matrix consisting of the N eigenvalues of \mathbf{X} and the columns of the real orthogonal matrix \mathbf{R} consist of the corresponding normalized eigenvectors. Using the notation of Definition 1.5, the product rule for differentiation gives

$$d\mathbf{X} = d\mathbf{R}\mathbf{L}\mathbf{R}^T + \mathbf{R}d\mathbf{L}\mathbf{R}^T + \mathbf{R}\mathbf{L}d\mathbf{R}^T.$$

Rather than take the wedge product of both sides of this equation, it is simpler to first premultiply by \mathbf{R}^T and postmultiply by \mathbf{R} to obtain

$$\begin{aligned}\mathbf{R}^T d\mathbf{X}\mathbf{R} &= \mathbf{R}^T d\mathbf{R}\mathbf{L} + \mathbf{L}d\mathbf{R}^T\mathbf{R} + d\mathbf{L} \\ &= \mathbf{R}^T d\mathbf{R}\mathbf{L} - \mathbf{L}\mathbf{R}^T d\mathbf{R} + d\mathbf{L},\end{aligned}\tag{1.29}$$

where to obtain the last line the formula $d\mathbf{R}^T\mathbf{R} = -\mathbf{R}^T d\mathbf{R}$ has been used (this follows from $\mathbf{R}\mathbf{R}^T = \mathbf{1}$).

According to Proposition 1.5

$$(\mathbf{R}^T d\mathbf{X}\mathbf{R}) = (\det \mathbf{R})^{N+1} (d\mathbf{X}).\tag{1.30}$$

But \mathbf{R} is an orthogonal matrix and so $\det \mathbf{R} = \pm 1$. As already noted, since only the modulus of J occurs in the change of variables formula, this sign factor can be ignored.

The wedge product of the r.h.s. of (1.29) can be taken with the aid of the following result.

Proposition 1.6 *With the notation $\vec{r}_k = (r_{1k}, r_{2k}, \dots, r_{Nk})^T$, for the k th column of \mathbf{R} we have*

$$\begin{aligned}\mathbf{R}^T d\mathbf{R}\mathbf{L} - \mathbf{L}\mathbf{R}^T d\mathbf{R} + d\mathbf{L} \\ = \begin{bmatrix} d\lambda_1 & (\lambda_2 - \lambda_1)\vec{r}_1^T \cdot d\vec{r}_2 & \dots & (\lambda_N - \lambda_1)\vec{r}_1^T \cdot d\vec{r}_N \\ (\lambda_2 - \lambda_1)\vec{r}_1^T \cdot d\vec{r}_2 & d\lambda_2 & \dots & (\lambda_N - \lambda_2)\vec{r}_2^T \cdot d\vec{r}_N \\ \vdots & \vdots & \ddots & \vdots \\ (\lambda_N - \lambda_1)\vec{r}_1^T \cdot d\vec{r}_N & (\lambda_N - \lambda_2)\vec{r}_2^T \cdot d\vec{r}_N & \dots & d\lambda_N \end{bmatrix}\end{aligned}$$

Proof This follows by explicitly forming the matrix products, and simplifying the resulting expression by noting from $\mathbf{R}\mathbf{R}^T = \mathbf{1}$ that $\vec{r}_j^T \cdot d\vec{r}_k = -\vec{r}_k \cdot d\vec{r}_j^T = -\vec{r}_k^T \cdot d\vec{r}_j$.

From Proposition 1.6 and Definition 1.6, the wedge product of the r.h.s. of (1.29) can be written down (note in particular that the matrix in Proposition 1.6 is symmetric), whereas (1.30) gives the wedge product of the l.h.s. of (1.29). Equating these expressions gives

$$(d\mathbf{X}) = \prod_{1 \leq j < k \leq N} (\lambda_k - \lambda_j) \bigwedge_{j=1}^N d\lambda_j (\mathbf{R}^T d\mathbf{R}).\tag{1.31}$$

The factorization property of the Jacobian between the eigenvalues and the variables involving the eigenvectors is evident and the expression (1.24) for the eigenvalue p.d.f. of the GOE follows. The p.d.f. for the components of the eigenvectors is calculated in Exercises 1.3 q.2.

1.3.3 Scaling of the Jacobian

Here we will show how the eigenvalue factor in the Jacobian can be deduced by considering a simple scaling property of the wedge product. Since there are $N(N+1)/2$ independent elements in \mathbf{X} , $(d\mathbf{X})$ consists of the product of $N(N+1)/2$ independent differentials. Thus if we multiply \mathbf{X} by a scalar a , we have that $(da\mathbf{X}) = a^{N(N+1)/2} (d\mathbf{X})$. On the other hand, with $\mathbf{X} = \mathbf{R}\mathbf{L}\mathbf{R}^T$, we know that $(\mathbf{R}\mathbf{L}\mathbf{R}^T)$ is a polynomial in $\lambda_1, \dots, \lambda_N$. Since $a\mathbf{X} = \mathbf{R}a\mathbf{L}\mathbf{R}^T$, the scaling property

of $(da\mathbf{X})$ gives that in fact $(\mathbf{R}\mathbf{L}\mathbf{R}^T)$ is a homogeneous polynomial of degree $N(N-1)/2$ (here we have subtracted N from $N(N+1)/2$ to account for the scaling of the measure $d\lambda_1 \cdots d\lambda_N$). Since the Jacobian must vanish for $\lambda_j = \lambda_k$, which is equivalent to saying that repeated eigenvalues occur with zero probability, the polynomial factor is necessarily proportional to $\prod_{j < k} (\lambda_k - \lambda_j)$, in agreement with the above calculation.

1.3.4 Metric forms

Another approach to deriving (1.31) is through the use of a metric form defined on the space of symmetric matrices [110]. For an $N \times N$ real symmetric matrix \mathbf{X} , the metric form of the line element ds is specified by

$$(ds)^2 = \text{Tr}(d\mathbf{X}d\mathbf{X}^T) = \sum_{j=1}^N (dx_{jj})^2 + 2 \sum_{j < k} (dx_{jk})^2, \quad (1.32)$$

(of course $d\mathbf{X}^T = d\mathbf{X}$, but it is convenient to write as presented), and the volume measure is

$$(d\mathbf{X}) = \bigwedge_{j \leq k} dx_{jk}.$$

If one now makes a change of variables, expressing the elements x_{jk} in terms of some new variables y_{jk} such that

$$(ds)^2 = \sum_{j=1}^N (h_{jj} dy_{jj})^2 + 2 \sum_{j < k} (h_{jk} dy_{jk})^2, \quad (1.33)$$

where the h_{jk} 's typically depend on $\{y_{jk}\}$, the corresponding volume measure is

$$(d\mathbf{X}) = \left(\bigwedge_{j \leq k} h_{jk} \right) (d\mathbf{Y}), \quad (1.34)$$

thus giving a change of variable formula for the volume measure.

More generally the metric forms method gives that if $(ds)^2$ is a symmetric quadratic form in some independent infinitesimals $\{dy_\mu\}$, so that

$$(ds)^2 = \sum_{\mu, \nu} g_{\mu, \nu} dy_\mu dy_\nu, \quad g_{\mu, \nu} = g_{\nu, \mu} \quad (1.35)$$

then the corresponding volume measure is

$$\left(\det[g_{\mu, \nu}] \right)^{1/2} \bigwedge_{\mu} dy_\mu. \quad (1.36)$$

Comparing (1.35) with (1.33), we see that there are only diagonal terms present in the formula for the line element. The determinant is then the product of the diagonal terms, which is consistent with (1.34).

We can apply this formalism by noting from (1.29) and Proposition 1.6 that

$$\text{Tr}(d\mathbf{X}d\mathbf{X}^T) = \sum_{j < k} (\lambda_k - \lambda_j)^2 (\vec{r}_j \cdot d\vec{r}_k)^2 + \sum_{j=1}^N (d\lambda_j)^2.$$

Application of (1.34) then reclaims (1.31).

Exercises 1.3

1.(i) Let \mathbf{R} be a $N \times N$ real orthogonal matrix. Show that in general \mathbf{R} has $N^2 - N(N-1)/2 - N$ independent elements.

(ii) Use (i) to show that the number of independent elements on both sides of the equation $\mathbf{X} = \mathbf{R}\mathbf{L}\mathbf{R}^T$, where \mathbf{X} is real symmetric and \mathbf{L} is diagonal, is equal.

2. Here, following [103], the distribution of the components of the eigenvectors in the GOE is calculated.

(i) Note that for matrices in the GOE every eigenvector can be transformed by an arbitrary real orthogonal matrix, and still remains an eigenvector of a matrix in the GOE. Conclude from this that the only invariant of the eigenvectors is their norm, and so the joint distribution of the components (u_1, \dots, u_N) is therefore given by

$$\frac{1}{C} \delta\left(1 - \sum_{p=1}^N u_p^2\right),$$

where $C = 2\pi^{N/2}/\Gamma(N/2)$ and represents the surface area of the unit N -sphere.

(ii) Show that the reduced, or marginal, joint distribution $p(u_1, \dots, u_n)$ obtained by integrating out the variables u_{n+1}, \dots, u_N is given by

$$p(u_1, \dots, u_n) = \pi^{-n/2} \frac{\Gamma(N/2)}{\Gamma((N-n)/2)} \left(1 - \sum_{p=1}^n u_p^2\right)^{(N-n-2)/2}.$$

For this purpose write the delta function in (i) as a Fourier integral.

(iii) From (ii) show that for large N ,

$$\frac{1}{N^{n/2}} p\left(\frac{u_1}{\sqrt{N}}, \dots, \frac{u_n}{\sqrt{N}}\right) \sim \left(\frac{2}{\pi}\right)^{n/2} e^{-\frac{1}{2} \sum_{p=1}^n u_p^2}. \quad (1.37)$$

(iv) Show that forming a vector (u_1, \dots, u_N) in which each component has distribution $x_j/(x_1^2 + \dots + x_N^2)^{1/2}$, with the x_j 's standard normal random variables, implies the vector is uniformly distributed on the unit N -sphere and thus has joint density as in (i). Use this fact to rederive (1.37).

3. (i) For a general 2×2 real symmetric matrix

$$\mathbf{A} := \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

show that the (unordered) eigenvalues are given by

$$\lambda_{\pm} = \frac{1}{2}(a+c) \pm \frac{1}{2}\left((a-c)^2 + 4b^2\right)^{1/2}.$$

(ii) For the matrix in (i) parametrize the matrix of eigenvectors as

$$\mathbf{R} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

and from the diagonalization equation $\mathbf{A} = \mathbf{R} \text{diag}[\lambda_+, \lambda_-] \mathbf{R}^T$, read off that

$$a = \lambda_+ \cos^2 \theta + \lambda_- \sin^2 \theta, \quad b = (\lambda_+ - \lambda_-) \cos \theta \sin \theta, \quad c = \lambda_+ \sin^2 \theta + \lambda_- \cos^2 \theta.$$

(iii) Deduce from (ii) that

$$J := \begin{vmatrix} \frac{\partial a}{\partial \lambda_+} & \frac{\partial b}{\partial \lambda_+} & \frac{\partial c}{\partial \lambda_+} \\ \frac{\partial a}{\partial \lambda_-} & \frac{\partial b}{\partial \lambda_-} & \frac{\partial c}{\partial \lambda_-} \\ \frac{\partial a}{\partial \theta} & \frac{\partial b}{\partial \theta} & \frac{\partial c}{\partial \theta} \end{vmatrix} = (\lambda_+ - \lambda_-)$$

and note that this is consistent with (1.31).

1.4 Random Hermitian and quaternion real matrices

Since most physical systems possess a time reversal symmetry, the GOE correctly describes statistical properties of the spectra of many quantum systems. Examples are the excitations of heavy nuclei and the spectrum of the hydrogen atom in a strong magnetic field (the latter system has a “non-conventional” time reversal symmetry, see Haake [103]). Nonetheless the considerations of time reversal symmetry of Section 1.2.1 indicate two further random matrix ensembles.

1.4.1 The Gaussian unitary ensemble

For a quantum system without time reversal symmetry the only constraint on the complex Hermitian matrix used to model the discrete portion of the energy spectrum is that two matrices related by a similarity transformation of unitary operators have the same joint p.d.f. for the elements. This requirement is satisfied by the following choice of matrix ensemble.

Definition 1.7 *A random Hermitian $N \times N$ matrix \mathbf{X} is said to belong to the Gaussian unitary ensemble (GUE) if the diagonal elements (which must be real) and the upper triangular elements $x_{jk} = u_{jk} + iv_{jk}$ are independently chosen with p.d.f.*

$$\frac{1}{\sqrt{\pi}}e^{-x_{jj}^2} \quad \text{and} \quad \frac{2}{\pi}e^{-2(u_{jk}^2 + v_{jk}^2)} = \frac{2}{\pi}e^{-2|x_{jk}|^2}.$$

respectively. Equivalently, the diagonal entries have distribution $\mathcal{N}[0, 1/\sqrt{2}]$, while the upper triangular elements have distribution $\mathcal{N}[0, 1/2] + i\mathcal{N}[0, 1/2]$.

From this definition the joint p.d.f. of all the independent elements is

$$P(\mathbf{X}) := \prod_{j=1}^N \frac{1}{\sqrt{\pi}}e^{-x_{jj}^2} \prod_{1 \leq j < k \leq N} \frac{2}{\pi}e^{-2|x_{jk}|^2} = A_N \prod_{j,k=1}^N e^{-|x_{jk}|^2} = A_N e^{-\text{Tr}(\mathbf{X}^2)}$$

where A_N is the normalization. The invariance $P(\mathbf{U}^{-1}\mathbf{X}\mathbf{U}) = P(\mathbf{X})$ for any unitary matrix \mathbf{U} follows immediately.

1.4.2 The Gaussian symplectic ensemble

In subsection 1.2.1 it was remarked that in quantum systems with a time reversal symmetry T , either $T^2 = 1$ or $T^2 = -1$ with $T = \mathbf{Z}_{2N}K$. Consideration of the former case leads to real symmetric matrices. Here the latter possibility will be discussed.

Now, since T commutes with the $2N \times 2N$ matrix \mathbf{X} modelling the Hamiltonian, \mathbf{X} must in addition to being Hermitian have the property

$$\mathbf{X} = T\mathbf{X}T^{-1} = \mathbf{Z}_{2N}K\mathbf{X}K^{-1}\mathbf{Z}_{2N}^{-1} = \mathbf{Z}_{2N}K\mathbf{X}K\mathbf{Z}_{2N}^{-1} = \mathbf{Z}_{2N}\bar{\mathbf{X}}\mathbf{Z}_{2N}^{-1}. \quad (1.38)$$

Since \mathbf{Z}_{2N} is block diagonal, with blocks (1.22), a $2N \times 2N$ matrix \mathbf{X} with this property can be viewed as an $N \times N$ matrix with elements consisting of 2×2 blocks of the form

$$\begin{bmatrix} z & w \\ -\bar{w} & \bar{z} \end{bmatrix}, \quad (1.39)$$

where z and w are complex numbers. A 2×2 matrix of this form is said to be *real quaternion*. From an abstract perspective the quaternions are an algebra with elements of the form

$$a_0 + a_1 i + a_2 j + a_3 k, \quad i^2 = j^2 = k^2 = -1, \quad ijk = -1, \quad (1.40)$$

where a_0, \dots, a_3 are scalars. The basis elements $1, i, j, k$ can be realized as 2×2 matrices with complex elements given by

$$\mathbf{1} := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \mathbf{e}_1 := i\boldsymbol{\sigma}_z = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \quad \mathbf{e}_2 := i\boldsymbol{\sigma}_y = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \mathbf{e}_3 := i\boldsymbol{\sigma}_x = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \quad (1.41)$$

respectively. Forming a general linear combination, consisting of real scalar multiples of these basis elements gives the structure (1.39).

For future reference we note that with a real quaternion \mathbf{q} written in the form $\mathbf{q} = c_0 \mathbf{1} + c_1 \mathbf{e}_1 + c_2 \mathbf{e}_2 + c_3 \mathbf{e}_3$ the dual, denoted $\bar{\mathbf{q}}$ or \mathbf{q}^D , is defined as

$$\bar{\mathbf{q}} = \mathbf{q}^D = c_0 \mathbf{1} - c_1 \mathbf{e}_1 - c_2 \mathbf{e}_2 - c_3 \mathbf{e}_3. \quad (1.42)$$

With this definition the dual of (1.39) is

$$\begin{bmatrix} \bar{z} & -w \\ \bar{w} & z \end{bmatrix}. \quad (1.43)$$

Furthermore, with $|\mathbf{q}|^2 := \bar{\mathbf{q}}\mathbf{q} = \mathbf{q}\bar{\mathbf{q}}$, we have $|\mathbf{q}|^2 = c_0^2 + c_1^2 + c_2^2 + c_3^2$, the relation $|\mathbf{q}_1 \mathbf{q}_2| = |\mathbf{q}_1| |\mathbf{q}_2|$ holds, and each non-zero \mathbf{q} has a unique inverse, $\mathbf{q}^{-1} = \bar{\mathbf{q}}/|\mathbf{q}|^2$.

An $N \times N$ matrix with real quaternion elements is said to be *quaternion real*. This structure motivates the definition of the third and final ensemble of Gaussian random matrices as motivated by quantum physics.

Definition 1.8 *A random Hermitian $N \times N$ matrix \mathbf{X} with real quaternion elements is said to belong to the Gaussian symplectic ensemble (GSE) if the elements z_{jj} of each diagonal real quaternion (which must be real) are independently chosen with p.d.f.*

$$\sqrt{\frac{2}{\pi}} e^{-2z_{jj}^2}$$

(or equivalently have distribution $\mathbb{N}[0, 1/2]$) while the upper triangular off diagonal elements $z_{jk} = u_{jk} + iv_{jk}$ and $w_{jk} = u'_{jk} + iv'_{jk}$ are independently chosen with real and imaginary parts having p.d.f.

$$\frac{4}{\pi} e^{-4|z_{jk}|^2} \quad \text{and} \quad \frac{4}{\pi} e^{-4|w_{jk}|^2}$$

(or equivalently have distribution $\mathbb{N}[0, 1/2\sqrt{2}] + i\mathbb{N}[0, 1/2\sqrt{2}]$).

A fundamental property of quaternion real Hermitian matrices, which follows from the first equation in (1.38), is that their spectrum is doubly degenerate (see Exercises 1.4 q.1).

It follows from Definition 1.8 that the joint p.d.f. of all the independent elements of the GSE is given by

$$P(\mathbf{X}) = A_N e^{-\text{Tr}(\mathbf{X}^2)},$$

where A_N denotes the normalization. This satisfies the general requirement of being invariant with respect to similarity transformations of appropriate unitary matrices. In fact the appropriate unitary matrices are those which under a similarity transformation map a quaternion real Hermitian matrix into another quaternion real Hermitian matrix. This subgroup of unitary matrices is specified by the following result.

Proposition 1.7

(a) Let \mathbf{X} be an arbitrary $N \times N$ Hermitian matrix with real quaternion elements, so that in general the only symmetry of \mathbf{X} (other than some multiple of the identity) is the operator $T = \mathbf{Z}_{2N}K$. Then any unitary matrix \mathbf{U} which under a similarity transformation maps \mathbf{X} into another Hermitian matrix with real quaternion elements must commute or anticommute with T .

(b) A unitary matrix \mathbf{U} which commutes with $T = \mathbf{Z}_{2N}K$ has the property

$$\mathbf{U}\mathbf{Z}_{2N}\mathbf{U}^T = \mathbf{Z}_{2N}, \tag{1.44}$$

which implies \mathbf{U} is equivalent to a symplectic matrix, while a unitary matrix \mathbf{U} which anticommutes with T has the property $-\mathbf{U}\mathbf{Z}_{2N}\mathbf{U}^T = \mathbf{Z}_{2N}$.

Proof (a) Let \mathbf{X}' be such that $\mathbf{U}^{-1}\mathbf{X}\mathbf{U} = \mathbf{X}'$. Since both \mathbf{X} and \mathbf{X}' are quaternion real, T commutes with both of these matrices. This implies $\mathbf{X}T\mathbf{U}T^{-1} = T\mathbf{U}T^{-1}\mathbf{X}'$. Comparing these two equations gives that $T\mathbf{U}T^{-1}\mathbf{U}^{-1}$ commutes with \mathbf{X} . But the only operators which commute with \mathbf{X} are T and some multiple of the identity, so the above combination of operators must equal one of these operators. We see that the first choice leads to $T = 1$ which is a contradiction, while the second gives $T\mathbf{U} = \pm\mathbf{U}T$ (regarding the signs, recall Exercises 1.2, q.4(ii)) as required.

(b) At the beginning of this subsection it was noted that any matrix, in this case \mathbf{U} , which commutes with T has the property $\mathbf{U} = \mathbf{Z}_{2N}\bar{\mathbf{U}}\mathbf{Z}_{2N}^{-1}$. The equation (1.44) follows after noting $\bar{\mathbf{U}} = (\mathbf{U}^{-1})^T$ and rearranging. Now, by definition a matrix is symplectic if

$$\mathbf{X}^T\mathbf{J}_{2N}\mathbf{X} = \mathbf{J}_{2N}, \quad \mathbf{J}_{2N} := \begin{bmatrix} \mathbf{0}_N & \mathbf{1}_N \\ -\mathbf{1}_N & \mathbf{0}_N \end{bmatrix}. \tag{1.45}$$

If \mathbf{X} is also unitary this implies \mathbf{X} has the block structure

$$\mathbf{X} = \begin{bmatrix} \mathbf{Z} & \mathbf{W} \\ -\bar{\mathbf{W}} & \mathbf{Z} \end{bmatrix}$$

(cf. (1.39)). Now, the matrix \mathbf{J}_{2N} is related to \mathbf{Z}_{2N} by a similarity transformation $\mathbf{J}_{2N} = \mathbf{Q}^{-1}\mathbf{Z}_{2N}\mathbf{Q}$ where \mathbf{Q} is a unitary matrix with elements ± 1 (there must therefore be exactly one non-zero element in each row/column). We thus conclude from (1.44) that $\mathbf{Q}^{-1}\mathbf{U}\mathbf{Q}$ is symplectic. The only difference in the anticommuting case is a minus sign, which gives the second result.

1.4.3 The eigenvalue p.d.f.'s

The calculation of the eigenvalue p.d.f.'s from the joint p.d.f.'s for the elements of the GUE and GSE can be done in a similar way to that presented in Section 1.3 for the GOE. The required

working is sketched in Exercises 1.4 q.3 and q.4, and the final results are summarized in the following result, which for completeness also contains the eigenvalue p.d.f. for the GOE.

Proposition 1.8 *Let \mathbf{H} be an Hermitian matrix with real ($\beta = 1$), complex ($\beta = 2$) and real quaternion ($\beta = 4$) elements, and let \mathbf{H} be decomposed in terms of its eigenvalues and eigenvectors via the formula $\mathbf{H} = \mathbf{U}\mathbf{L}\mathbf{U}^\dagger$ where \mathbf{L} is a diagonal matrix consisting of the eigenvalues of \mathbf{H} , and \mathbf{U} is a unitary matrix with real ($\beta = 1$), complex ($\beta = 2$) and real quaternion ($\beta = 4$) elements consisting of the corresponding eigenvectors. We have*

$$(d\mathbf{H}) = \prod_{1 \leq j < k \leq N} |\lambda_k - \lambda_j|^\beta \bigwedge_{j=1}^N d\lambda_j (\mathbf{U}^\dagger d\mathbf{U}), \quad (1.46)$$

and hence for an appropriate choice of the normalization $G_{\beta N}$, which is given explicitly by (1.76),

$$\frac{1}{G_{\beta N}} \exp\left(-\frac{\beta}{2} \sum_{j=1}^N \lambda_j^2\right) \prod_{1 \leq j < k \leq N} |\lambda_k - \lambda_j|^\beta, \quad (1.47)$$

with $\beta = 1, 2$ and 4 is the eigenvalue p.d.f. for the GOE, GUE and GSE respectively.

1.4.4 Relationship to Lie algebras

The sets of matrices

$$\begin{aligned} gl(N, \mathbf{R}) &:= \{\text{all } N \times N \text{ real matrices}\} \\ gl(N, \mathbf{C}) &:= \{\text{all } N \times N \text{ complex matrices}\} \\ u^*(2N) &:= \{\text{all } N \times N \text{ real quaternion matrices}\} \end{aligned}$$

are all closed under commutation and so form matrix Lie algebras.

Now, in general a matrix can be decomposed as the sum of a Hermitian and an anti-Hermitian matrix. We see that the Hermitian component of the above Lie algebras corresponds to Hermitian matrices with real, complex and real quaternion elements respectively. This is significant for a number of reasons. One is from a classification perspective. In fact there are only ten distinct infinite families of matrix Lie algebras. We will see that each of the remaining seven cases also occur in a basic quantum mechanics problem constrained by a global symmetry. Furthermore, the identification with matrix Lie algebras implies a one-to-one correspondence between the ten families of Hermitian matrices and the ten families of unitary matrices. This comes about because of the relationship between matrix Lie algebras and symmetric spaces. To each matrix Lie algebra there corresponds a non-compact and compact symmetric space, with the former being isomorphic to a certain set of Hermitian matrices, and the latter isomorphic to a certain set of unitary matrices. Some more details are given in Section 2.1.1. The isomorphism with symmetric spaces has the consequence that the eigenvalue dependent portion of the Jacobian in $(d\mathbf{H})$ can be written in the form

$$\prod_{\vec{\alpha} \in R_+} |\langle \vec{\alpha}, \vec{\lambda} \rangle|^{m_\alpha}$$

where $\vec{\alpha}$, an N component Euclidean vector, is a so called root of the root system corresponding to the symmetric space, \langle , \rangle is the dot product, R_+ is the set of positive roots, and m_α the

multiplicity of $\vec{\alpha}$. This structure, in the case of the symmetric spaces corresponding to the classical groups, appears in the so called *Weyl integration formula* [210]. For the symmetric spaces corresponding to the Gaussian ensembles the positive roots are $\vec{e}_j - \vec{e}_k$ ($j < k$) (root system of type A — see Section 3.5.1) with multiplicities $m_\alpha = \beta$, and this reclaims the eigenvalue dependent portion of (1.46). However we will not pursue the derivation of these facts, which can be found in [107].

1.4.5 Octonions and the $N = 2, \beta = 8$ Gaussian ensemble

The p.d.f. (1.47) for $N = 2, \beta = 8$ can be realized as the eigenvalues of a random Hermitian matrix with real octonion elements. To see this, we must first revise aspects of the theory of real octonions [193]. The real octonions can be constructed out of the real quaternions. Let p_1, p_2, q_1, q_2 be abstract real quaternions, and thus linear combinations with real coefficients of $\{1, i, j, k\}$ as specified by (1.40). Let \bar{q} denote the quaternionic dual defined by (1.42), and let l denote a quantity algebraically distinct from the real quaternions. The real octonion algebra then consists of elements of the form $a = p_1 + p_2l, b = q_1 + q_2l$, with addition and multiplication defined by

$$a + b = (p_1 + q_1) + (p_2 + q_2)l, \quad ab = (p_1q_1 - \bar{q}_2p_2) + (q_2p_1 + p_2\bar{q}_1)l \quad (1.48)$$

respectively. It follows that the real octonions are an eight dimensional algebra with basis

$$1, e_1 := i, e_2 := j, e_3 := k, e_4 := l, e_5 := il, e_6 := jl, e_7 := kl.$$

In general

$$a(bc) \neq (ab)c$$

(for example with $a = e_5, b = e_6, c = e_7$ we have $a(bc) = -e_4$ and $(ab)c = e_4$), so unlike the real quaternions the real octonions are not associative. On the other hand, with $\bar{a} := \bar{p}_1 - p_2l, \bar{p}_1$ denoting the quaternionic dual (1.42), we have $\overline{ab} = \bar{b}\bar{a}$ and thus with $|a| := \sqrt{a\bar{a}} = \sqrt{\bar{a}a}$,

$$|ab| = |a||b|. \quad (1.49)$$

Furthermore, with a general real octonion written as $a = a_0 + a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4 + a_5e_5 + a_6e_6 + a_7e_7$, we have

$$|a| = \sqrt{a_0^2 + a_1^2 + \cdots + a_7^2}, \quad (1.50)$$

and it is also true that each $a \neq 0$ has a unique inverse specified by

$$a^{-1} = \bar{a}/(\bar{a}a). \quad (1.51)$$

The properties (1.49)–(1.51) say that the real octonions are a normed division algebra. In fact a theorem of Hurwitz [111] says that up to isomorphisms, the only normed division algebras over the reals, with a unit element, are the reals, complex numbers, real quaternions and the real octonions.

Because the real octonions are not associative, they cannot be represented as a matrix algebra. Nonetheless, the actions of right and left multiplication by a given real octonion a on a general real octonion x can be represented as a matrix. To specify these matrices, we first require the corresponding result for the real quaternions, which follows immediately from the corresponding multiplication rule.

Proposition 1.9 *Let $x = x_0 + x_1i + x_2j + x_3k$ be a real quaternion and let $\vec{x} = (x_0, x_1, x_2, x_3)^T$ denote the column vector formed from the coefficients. Then*

$$\vec{a}x = \phi(a)\vec{x}, \quad \vec{x}a = \tau(a)\vec{x}$$

where

$$\phi(a) = \begin{bmatrix} a_0 & -a_1 & -a_2 & -a_3 \\ a_1 & a_0 & -a_3 & a_2 \\ a_2 & a_3 & a_0 & -a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{bmatrix}, \quad \tau(a) = \mathbf{K}\phi^T(a)\mathbf{K} = \begin{bmatrix} a_0 & -a_1 & -a_2 & -a_3 \\ a_1 & a_0 & a_3 & -a_2 \\ a_2 & -a_3 & a_0 & a_1 \\ a_3 & a_2 & -a_1 & a_0 \end{bmatrix},$$

with $\mathbf{K} = \text{diag}[1, -1, -1, -1]$.

Using Proposition 1.9, the corresponding result for the real octonions follows from the multiplication rule (1.48).

Proposition 1.10 *Let $x = x_0 + x_1e_1 + \dots + x_7e_7$ be a real octonion, and let $\vec{x} = (x_0, x_1, \dots, x_7)^T$ denote the column vector formed from the coefficients. Then with $a = a_1 + a_2l$ a real octonion, and thus a_1, a_2 real quaternions, and $\tilde{\mathbf{K}} = \text{diag}[\mathbf{K}, \mathbf{1}_4]$ we have*

$$\vec{a}x = \omega(a)\vec{x}, \quad \vec{x}a = \nu(a)\vec{x}$$

where

$$\omega(a) = \begin{bmatrix} \phi(a_1) & -\tau(a_2)\mathbf{K} \\ \phi(a_2)\mathbf{K} & \tau(a_1) \end{bmatrix} = \begin{bmatrix} a_0 & -a_1 & -a_2 & -a_3 & -a_4 & -a_5 & -a_6 & -a_7 \\ a_1 & a_0 & -a_3 & a_2 & -a_5 & a_4 & a_7 & -a_6 \\ a_2 & a_3 & a_0 & -a_1 & -a_6 & -a_7 & a_4 & a_5 \\ a_3 & -a_2 & a_1 & a_0 & -a_7 & a_6 & -a_5 & a_4 \\ a_4 & a_5 & a_6 & a_7 & a_0 & -a_1 & -a_2 & -a_3 \\ a_5 & -a_4 & a_7 & -a_6 & a_1 & a_0 & a_3 & -a_2 \\ a_6 & -a_7 & -a_4 & a_5 & a_2 & -a_3 & a_0 & a_1 \\ a_7 & a_6 & -a_5 & -a_4 & a_3 & a_2 & -a_1 & a_0 \end{bmatrix}$$

$$\nu(a) = \tilde{\mathbf{K}}\omega^T(a)\tilde{\mathbf{K}}.$$

Consider now the 2×2 Hermitian matrix with real octonion entries

$$\mathbf{A} = \begin{bmatrix} a & b \\ \bar{b} & c \end{bmatrix}.$$

For \mathbf{A} to be Hermitian, the elements a and c must in fact be real, and thus

$$\omega(\mathbf{A}) = \begin{bmatrix} a\mathbf{1}_8 & \omega(b) \\ \omega^T(b) & c\mathbf{1}_8 \end{bmatrix}. \quad (1.52)$$

Adding together appropriate (octonion) multiples of rows and columns shows that this matrix is similar to the matrix

$$\begin{bmatrix} a\mathbf{1}_8 & \mathbf{1}_4 \otimes \begin{bmatrix} b & 0 \\ 0 & \bar{b} \end{bmatrix} \\ \mathbf{1}_4 \otimes \begin{bmatrix} \bar{b} & 0 \\ 0 & b \end{bmatrix} & c\mathbf{1}_8 \end{bmatrix}$$

and thus the characteristic polynomial is given by

$$\det(\boldsymbol{\omega}(\mathbf{A}) - \lambda \mathbf{1}_{16}) = \left((a - \lambda)(c - \lambda) - b\bar{b} \right)^8.$$

This shows each eigenvalue is eight fold degenerate.

Regarding the eigenvectors, as the number of independent real elements in (1.52) is ten, and there are two distinct eigenvalues, there are a total of eight independent components. This implies that in the analogue of Proposition 1.6, exactly eight components are to be multiplied together in any one term, and consequently

$$(d\mathbf{A}) = (\lambda_1 - \lambda_2)^8 d\lambda_1 d\lambda_2 (\mathbf{U}^\dagger d\mathbf{U}). \quad (1.53)$$

Furthermore, choosing the elements a, c and the components b_j of $\boldsymbol{\omega}(b)$ in (1.52) to have the Gaussian distributions

$$\frac{2}{\sqrt{\pi}} e^{-4a^2}, \quad \frac{2}{\sqrt{\pi}} e^{-4c^2}, \quad \sqrt{\frac{8}{\pi}} e^{-8b_j^2}$$

respectively, we then have that the joint distribution of the independent elements is proportional to

$$e^{-\text{Tr}((\boldsymbol{\omega}(\mathbf{A}))^2)/2}.$$

This together with (1.53) implies that the eigenvalue p.d.f. is given by (1.47) with $N = 2$, $\beta = 8$.

Exercises 1.4

1. The aim of this exercise is to show that if a $2N \times 2N$ Hermitian matrix \mathbf{X} commutes with the time reversal operator $T = \mathbf{Z}_{2N} K$, then the eigenvalues of \mathbf{X} are doubly degenerate (this is known as Kramer's degeneracy).

(i) Suppose $|\phi\rangle$ is an eigenvector of \mathbf{X} with eigenvalue λ . State why $T|\phi\rangle$ is also an eigenvector with eigenvalue λ .

(ii) Use the facts that T satisfies the formula of Exercises 1.2 q.3(i) and $T^2 = -1$ to show that $\langle \phi | T\phi \rangle = 0$ and hence deduce the desired result.

2. Let \mathbf{A} and \mathbf{M} be $N \times N$ matrices, where \mathbf{A} is non-singular. In this exercise, following [145, pg. 32] it will be shown that for \mathbf{A} real ($\beta = 1$), complex ($\beta = 2$) and real quaternion ($\beta = 4$), and \mathbf{M} real symmetric ($\beta = 1$), Hermitian ($\beta = 2$) and quaternion real Hermitian ($\beta = 4$),

$$(\mathbf{A}^\dagger d\mathbf{M}\mathbf{A}) = \left(\det(\mathbf{A}^\dagger \mathbf{A}) \right)^{\beta(N-1)/2+1} (d\mathbf{M}), \quad (1.54)$$

up to a \pm sign. In the case $\beta = 1$, this is the statement of Proposition 1.5. The idea is to decompose \mathbf{A} in terms of elementary matrices $\mathbf{A} = \mathbf{E}_p \mathbf{E}_{p-1} \cdots \mathbf{E}_1$. Each elementary matrix is either a permutation matrix $\mathbf{E}^{(j \leftrightarrow k)}$ (the identity matrix with rows j and k interchanged), a matrix $\mathbf{E}^{(j \rightarrow \alpha j)}$ which multiplies row j by the constant α with α real ($\beta = 1$), complex ($\beta = 2$) or real quaternion ($\beta = 4$) (the identity matrix with row j multiplied by α), or the matrix $\mathbf{E}^{(j \rightarrow j+k)}$ which adds together two rows (the identity matrix with row k added to row j).

(i) Show by explicit calculation that for any matrix \mathbf{X} of the same type as \mathbf{M}

$$(\mathbf{E}^{(j \leftrightarrow k)} d\mathbf{X} \mathbf{E}^{(j \leftrightarrow k)^\dagger}) = (d\mathbf{X}), \quad (\mathbf{E}^{(j \rightarrow j+k)} d\mathbf{X} \mathbf{E}^{(j \rightarrow j+k)^\dagger}) = (d\mathbf{X})$$

while, up to a \pm sign

$$(\mathbf{E}^{(j \rightarrow \alpha j)} d\mathbf{X} \mathbf{E}^{(j \rightarrow \alpha j)^\dagger}) = |\alpha|^{\beta(N-1)+2} (d\mathbf{X}) = |\det \mathbf{E}^{(j \rightarrow \alpha j)}|^{\beta(N-1)+2} (d\mathbf{X}).$$

(ii) Use the result of (i) to deduce the stated result.

For printing purposes, the symbol α^* rather than $\bar{\alpha}$ is used in the exercises below to denote the complex conjugate.

3. The aim of this exercise is to calculate the change of variables from the independent elements of an Hermitian matrix \mathbf{X} to the eigenvalues $\lambda_1, \dots, \lambda_N$ and other independent variables.

(i) From the diagonalization formula $\mathbf{X} = \mathbf{U}\mathbf{L}\mathbf{U}^{-1}$ where $\mathbf{L} := \text{diag}[\lambda_1, \dots, \lambda_N]$ and \mathbf{U} is a unitary matrix with columns given by the eigenvectors of \mathbf{X} , show that

$$(\mathbf{U}^{-1}d\mathbf{X}\mathbf{U}) = (\mathbf{U}^{-1}d\mathbf{U}\mathbf{L} - \mathbf{L}\mathbf{U}^{-1}d\mathbf{U} + d\mathbf{L})$$

and write down a formula for the Jacobian in terms of $(d\mathbf{X})$. Use the result of q.2 to show that the l.h.s. is equal to $(d\mathbf{X})$.

(ii) Show that $\mathbf{U}^{-1}d\mathbf{U}\mathbf{L} - \mathbf{L}\mathbf{U}^{-1}d\mathbf{U} + d\mathbf{L}$ equals

$$\begin{bmatrix} d\lambda_1 & (\lambda_2 - \lambda_1)\vec{u}_1^\dagger \cdot d\vec{u}_2 & \dots & (\lambda_N - \lambda_1)\vec{u}_1^\dagger \cdot d\vec{u}_N \\ (\lambda_2 - \lambda_1)(\vec{u}_1^\dagger \cdot d\vec{u}_2)^* & d\lambda_2 & \dots & (\lambda_N - \lambda_2)\vec{u}_2^\dagger \cdot d\vec{u}_N \\ \vdots & \vdots & \ddots & \vdots \\ (\lambda_N - \lambda_1)(\vec{u}_1^\dagger \cdot d\vec{u}_N)^* & (\lambda_N - \lambda_2)(\vec{u}_2^\dagger \cdot d\vec{u}_N)^* & \dots & d\lambda_N \end{bmatrix}$$

(iii) Use the facts that $\vec{u}_j^\dagger \cdot d\vec{u}_k$ has independent real and imaginary parts and that only the elements on and above the diagonal are independent to conclude that the wedge product of the independent elements of the matrix in (ii) equals

$$\prod_{1 \leq j < k \leq N} (\lambda_k - \lambda_j)^2 \bigwedge_{j=1}^N d\lambda_j (\mathbf{U}^\dagger d\mathbf{U}).$$

(iv) Show that the factor dependent on the λ_j 's is consistent with the form required by the scaling $\mathbf{X} \mapsto a\mathbf{X}$ (recall Section 1.3.3).

4. Here the objective is the same as in q.2 above, except \mathbf{X} is now a $N \times N$ Hermitian matrix with real quaternion elements.

(i) From the diagonalization formula $\mathbf{X} = \mathbf{U}\mathbf{L}\mathbf{U}^{-1}$ where $\mathbf{L} = \text{diag}[\lambda_1 \mathbf{1}_2, \dots, \lambda_N \mathbf{1}_2]$ and \mathbf{U} is an $N \times N$ unitary matrix with real quaternion elements, write down the formulas analogous to those in q.3(i) and q.3(ii) above. To write down the analogue of (ii) use a matrix notation for the quaternion elements,

$$\vec{u}_k = (\mathbf{u}_{1k}, \dots, \mathbf{u}_{Nk})^T, \quad \vec{u}_j^\dagger \cdot \vec{u}_k = \sum_{p=1}^N \mathbf{u}_{pj}^\dagger \mathbf{u}_{pk}.$$

(ii) Use the facts that $\vec{u}_j^\dagger \cdot d\vec{u}_k$ has four independent terms, corresponding to the real and imaginary parts of the two independent terms in each real quaternion element, to deduce the formula analogous to q.3(iii) above. Also repeat the scaling analysis of q.3(iv) above.

5. An Hermitian matrix with zero real part is antisymmetric.

(i) Show that the non-zero eigenvalues of antisymmetric Hermitian matrices come in \pm pairs, λ_j and $-\lambda_j$ say, with corresponding eigenvectors $\vec{\phi}_j$ and $\vec{\phi}_j^*$, and that for N odd $\lambda = 0$ is an eigenvalue.

(ii) Use (i) to deduce that the equation of q.3(ii) holds with $\vec{u}_j = \vec{\phi}_j$, $\lambda_{N/2+j} = -\lambda_j$, $\vec{u}_{N/2+j} = \vec{u}_j^*$ ($j = 1, \dots, N/2$) N even, and $\vec{u}_N^* = \vec{u}_N = \phi_0$, $\lambda_N = 0$, $\vec{u}_j = \vec{\phi}_j$, $\lambda_{(N-1)/2+j} = -\lambda_j$ ($j = 1, \dots, (N-1)/2$) N odd. Use the fact that eigenvectors of an Hermitian matrix corresponding to distinct eigenvalues are orthogonal to deduce that $\vec{\phi}_0^\dagger \cdot d\vec{\phi}_j^* = 0$ ($j \neq 0$), and note too that $\vec{\phi}_0^\dagger \cdot d\vec{\phi}_k$ and $\vec{\phi}_0^{\dagger*} \cdot d\vec{\phi}_j^*$ are not independent.

(iii) Use (ii) to show that for an antisymmetric Hermitian $N \times N$ matrix \mathbf{H}^i , diagonalized by $\mathbf{H}^i = \mathbf{U}\mathbf{L}\mathbf{U}^{-1}$,

$$\begin{aligned} (d\mathbf{H}^i) &= \prod_{1 \leq j < k \leq N/2} (\lambda_j^2 - \lambda_k^2)^2 \bigwedge_{j=1}^{N/2} d\lambda_j (\mathbf{U}^\dagger d\mathbf{U}), \quad N \text{ even} \\ (d\mathbf{H}^i) &= \prod_{j=1}^{(N-1)/2} \lambda_j^2 \prod_{1 \leq j < k \leq (N-1)/2} (\lambda_j^2 - \lambda_k^2)^2 \bigwedge_{j=1}^{(N-1)/2} d\lambda_j (\mathbf{U}^\dagger d\mathbf{U}), \quad N \text{ odd.} \end{aligned}$$

(iv) Conclude from the result of (iii) that for a random antisymmetric Hermitian $N \times N$ matrix with upper triangular elements ix_{jk} chosen with p.d.f. $\sqrt{1/\pi}e^{-x_{jk}^2}$, the eigenvalue p.d.f. is equal to

$$\begin{aligned} \frac{1}{C} \prod_{j=1}^{N/2} e^{-\lambda_j^2} \prod_{1 \leq j < k \leq N/2} (\lambda_j^2 - \lambda_k^2)^2, \quad N \text{ even} \\ \frac{1}{C} \prod_{j=1}^{(N-1)/2} \lambda_j^2 e^{-\lambda_j^2} \prod_{1 \leq j < k \leq (N-1)/2} (\lambda_j^2 - \lambda_k^2)^2, \quad N \text{ odd.} \end{aligned}$$

6.(i) Let \mathbf{Q}^r be a quaternion real Hermitian matrix in which all entries are real, and let \mathbf{H} be the Hermitian matrix formed by replacing each quaternion element (1.39) by the scalar $z + iw$. Show that \mathbf{Q}^r and \mathbf{H} have the same distinct eigenvalues, and that the eigenvalues of \mathbf{Q}^r are doubly degenerate with eigenvectors of the form $\psi = \begin{bmatrix} \phi_k^{(r)} \\ \pm \phi_k^{(i)} \end{bmatrix}_{k=1, \dots, N}$ where $\phi = [\phi_k^{(r)} + i\phi_k^{(i)}]_{k=1, \dots, N}$ is an eigenvector of \mathbf{H} . Hence write down the eigenvalue p.d.f. of \mathbf{Q}^r .

(ii) Let \mathbf{Q} be a $N \times N$ real quaternion Hermitian matrix. Proceed in a converse fashion to (i) to write down a $4N \times 4N$ real symmetric matrix \mathbf{R} such that \mathbf{Q} and \mathbf{R} have the same distinct eigenvalues, and thus the same eigenvalue p.d.f. Also, relate the corresponding eigenvectors. Similarly, for \mathbf{H} a $N \times N$ complex Hermitian matrix, replace each entry $x + iy$ by its 2×2 real matrix representation

$$\begin{bmatrix} x & y \\ -y & x \end{bmatrix} \quad (1.55)$$

to obtain a doubly degenerate $2N \times 2N$ matrix for which the distinct eigenvalues coincide with those of \mathbf{H} .

7. Consider a quaternion real Hermitian matrix \mathbf{Q}^i in which all entries are pure imaginary so that \mathbf{Q}^i is antisymmetric.

(i) With the pair of eigenvectors corresponding to the doubly degenerate eigenvalues λ_j denoted by $\vec{\mathbf{u}}_j$ as in q.4, note from the theory of q.5(i) that $\vec{\mathbf{u}}_j^*$ is equal to the pair of eigenvectors corresponding to the doubly degenerate eigenvalue $-\lambda_j$.

(ii) Use the fact that eigenvectors of an Hermitian matrix corresponding to distinct eigenvalues are orthogonal to deduce that

$$\vec{\mathbf{u}}_j^\dagger \cdot \vec{\mathbf{u}}_j^* := \sum_{p=1}^N \mathbf{u}_{pj}^\dagger \mathbf{u}_{pj}^* = \sum_{p=1}^N \begin{bmatrix} 0 & -2\text{Im}(z_{pj}w_{pj}) \\ -2\text{Im}(z_{pj}w_{pj}) & 0 \end{bmatrix}, \quad \mathbf{u}_{jp} := \begin{bmatrix} z_{pj} & w_{pj} \\ -w_{pj}^* & z_{pj}^* \end{bmatrix},$$

and conclude from this that $\vec{\mathbf{u}}_j^\dagger \cdot d\vec{\mathbf{u}}_j^*$ has only one independent component. Note too that $\vec{\mathbf{u}}_j^\dagger \cdot d\vec{\mathbf{u}}_k$ and $\vec{\mathbf{u}}_j^* \cdot d\vec{\mathbf{u}}_k^*$ are not independent.

(iii) With the analogue of the equation of q.3(ii) in the quaternion case modified as in the first sentence of q.5(ii) (N even case), show from (i) that for an antisymmetric $N \times N$ quaternion real Hermitian matrix

Q^i diagonalized by $Q^i = \mathbf{U} \mathbf{L} \mathbf{U}^{-1}$,

$$\begin{aligned} (dQ^i) &= \prod_{j=1}^{N/2} |\lambda_j| \prod_{1 \leq j < k \leq N/2} (\lambda_j^2 - \lambda_k^2)^4 \bigwedge_{j=1}^{N/2} d\lambda_j (\mathbf{U}^\dagger d\mathbf{U}), \quad N \text{ even} \\ (dQ^i) &= \bigwedge_{j=1}^{(N-1)/2} |\lambda_j|^5 \prod_{1 \leq j < k \leq (N-1)/2} (\lambda_j^2 - \lambda_k^2)^4 \prod_{j=1}^{(N-1)/2} d\lambda_j (\mathbf{U}^\dagger d\mathbf{U}), \quad N \text{ odd.} \end{aligned}$$

8. [51] Let Q be a quaternion real matrix with the property that iQ is Hermitian.

(i) Note that Q must anticommute with the time reversal operator $T = \mathbf{Z}_{2N} K$, and use this to show that if $|\phi\rangle$ is an eigenvector of Q with eigenvalue λ , then $T|\phi\rangle$ is an eigenvector with eigenvalue $-\lambda$.

(ii) Proceed as in q.4 to show that with iQ diagonalized by $iQ = \mathbf{U} i \mathbf{L} \mathbf{U}^\dagger$, where

$$\mathbf{L} = \text{diag}(\lambda_1, -\lambda_1, \dots, \lambda_N, -\lambda_N),$$

and \mathbf{U} is unitary with real quaternion elements in which all elements are real,

$$(dQ) = \prod_{j=1}^N (2\lambda_j)^2 \prod_{1 \leq j < k \leq N} (\lambda_k^2 - \lambda_j^2)^2 \bigwedge_{j=1}^N d\lambda_j (\mathbf{U}^\dagger d\mathbf{U}). \quad (1.56)$$

1.5 Gaussian β -ensemble

A familiar technique in numerical linear algebra is the similarity transformation of a real symmetric matrix to tridiagonal form using a sequence of similarity transformations involving reflection matrices, referred to as Householder transformations. Explicitly, let \mathbf{A} be a real symmetric matrix $[a_{ij}]_{i,j=1,\dots,N}$. Then one can construct a sequence of real orthogonal matrices $\mathbf{U}^{(1)}, \mathbf{U}^{(2)}, \dots, \mathbf{U}^{(N-2)}$ such that

$$\mathbf{B}^{(N-2)} = \mathbf{U}^{(N-2)T} \mathbf{U}^{(N-3)T} \dots \mathbf{U}^{(1)T} \mathbf{A} \mathbf{U}^{(1)} \mathbf{U}^{(2)} \dots \mathbf{U}^{(N-2)} \quad (1.57)$$

where $\mathbf{B}^{(N-2)}$ is a symmetric tridiagonal matrix, and

$$\mathbf{U}^{(j)} = \mathbf{1} - 2\vec{u}^{(j)} \vec{u}^{(j)T} = \begin{bmatrix} \mathbf{1}_{j \times j} & \mathbf{0}_{j \times N-j} \\ \mathbf{0}_{N-j \times j} & \mathbf{V}_{N-j \times N-j} \end{bmatrix}, \quad (1.58)$$

where $\vec{u}^{(j)T} \vec{u}^{(j)} = 1$ and $\mathbf{V}_{N-j \times N-j}$ is real orthogonal (geometrically $\mathbf{U}^{(j)}$ corresponds to a reflection in the hyperplane orthogonal to $\vec{u}^{(j)}$).

Consider first the construction of $\mathbf{U}^{(1)}$. Choosing the components $u_l^{(1)}$ of $\vec{u}^{(1)}$ as

$$u_1^{(1)} = 0, \quad u_2^{(1)} = \left[\frac{1}{2} \left(1 - \frac{a_{12}}{\alpha} \right) \right]^{1/2}, \quad u_l^{(1)} = -\frac{a_{1l}}{2\alpha u_2^{(1)}}, \quad (l \geq 3) \quad (1.59)$$

where $\alpha = (a_{12}^2 + \dots + a_{1N}^2)^{1/2}$, one can readily check that

$$\mathbf{B}^{(1)} := \mathbf{U}^{(1)T} \mathbf{A} \mathbf{U}^{(1)} \quad (1.60)$$

has

$$b_{11} = a_{11}, \quad b_{12} = b_{21} = \alpha, \quad b_{1k} = b_{k1} = 0 \quad (k \geq 3)$$

and is thus tridiagonal with respect to the first row and column. The matrices $\mathbf{U}^{(j)}$, $j = 2, 3, \dots$ in order are now defined by the formulas (1.59), but with $u_1^{(j)} = u_2^{(j)} = \dots = u_j^{(j)} = 0$, and the analogue of the entries a_{1l} replaced by the elements in the first row of the bottom right $N - j \times N - j$ submatrix of $\mathbf{B}^{(j)}$.

Trotter [201], and Dumitriu and Edelman [50], posed the question as to the form of $\mathbf{B}^{(N-2)}$ when \mathbf{A} is a member of the GOE. It was found that like \mathbf{A} itself, the elements of $\mathbf{B}^{(N-2)}$ are all independent (apart from the requirement that $\mathbf{B}^{(N-2)}$ be symmetric) with a distribution that can be calculated explicitly [201].

Proposition 1.11 *Let $N[0, 1]$ refer to the standard normal distribution as defined below Definition 1.1, and let $\tilde{\chi}_k$ denote the square root of the gamma distribution $\Gamma[k/2, 1]$, the latter being specified by the density $(1/\Gamma(k/2))u^{k/2-1}e^{-u}$, $u > 0$, and realized by the sum of the squares of k independent Gaussian distributions $N[0, 1/\sqrt{2}]$. (The density of $\tilde{\chi}_k$ is thus equal to $(2/\Gamma(k/2))u^{k-1}e^{-u^2}$, $u > 0$.) Then for \mathbf{A} a member of the GOE, the tridiagonal matrix $\mathbf{B}^{(N-2)}$ obtained by successive Householder transformations is given by*

$$\begin{bmatrix} N[0, 1] & \tilde{\chi}_{N-1} & & & & \\ \tilde{\chi}_{N-1} & N[0, 1] & \tilde{\chi}_{N-2} & & & \\ & \tilde{\chi}_{N-2} & N[0, 1] & \tilde{\chi}_{N-3} & & \\ & & \ddots & \ddots & \ddots & \\ & & & \tilde{\chi}_2 & N[0, 1] & \tilde{\chi}_1 \\ & & & & \tilde{\chi}_1 & N[0, 1] \end{bmatrix}.$$

Proof From the Householder algorithm, the first row and column of $\mathbf{B}^{(N-2)}$ is the same as that of $\mathbf{B}^{(1)}$ in (1.60), and thus from (1.59) we have

$$b_{11}^{(N-2)} = N[0, 1], \quad b_{12}^{(N-2)} = \tilde{\chi}_{N-1},$$

where use has been made of the assumption that \mathbf{A} is a member of the GOE. To proceed further we must compute the distribution of the bottom $N - 1 \times N - 1$ block of $\mathbf{B}^{(1)}$. In general, denoting such a block of the matrix \mathbf{X} by \mathbf{X}_{N-1} , it follows from (1.58) that $\mathbf{B}_{N-1}^{(1)} = \mathbf{V}_{N-1}^T \mathbf{A}_{N-1} \mathbf{V}_{N-1}$. Since the elements of the real orthogonal matrix \mathbf{V}_{N-1} are independent of the elements of \mathbf{A}_{N-1} , which is a $N - 1 \times N - 1$ GOE matrix, it follows immediately from the general invariance of the GOE under orthogonal transformations that $\mathbf{B}_{N-1}^{(1)}$ is also a $N - 1 \times N - 1$ GOE matrix. Applying the Householder transformation to $\mathbf{B}_{N-1}^{(1)}$, we thus get

$$b_{22}^{(N-2)} = N[0, 1], \quad b_{23}^{(N-2)} = \tilde{\chi}_{N-2}.$$

Continuing inductively gives the stated result.

The result of Proposition 1.11 suggests investigating the Jacobian for the change of variables from a general tridiagonal matrix

$$\mathbf{T} = \begin{bmatrix} a_n & b_{n-1} & & & & \\ b_{n-1} & a_{n-1} & b_{n-2} & & & \\ & b_{n-2} & a_{n-2} & b_{n-3} & & \\ & & \ddots & \ddots & \ddots & \\ & & & b_2 & a_2 & b_1 \\ & & & & b_1 & a_1 \end{bmatrix}, \quad (1.61)$$

to its eigenvalues and variables relating to its eigenvectors. First, for each eigenvalue λ_k and corresponding eigenvector \vec{v}_k , it is easy to see by direct substitution that once the 1st component $\vec{v}_k^{(1)} =: q_k$ of \vec{v}_k is specified, all other components can be expressed in terms of λ_k and the elements of \mathbf{T} . To make the eigendecomposition unique we specify that $q_k > 0$, and furthermore note that \mathbf{T} , being symmetric, can be orthogonally diagonalized and so doing this we have

$$\sum_{k=1}^n q_k^2 = 1.$$

The Jacobian for the change of variables from

$$\vec{a} := (a_n, a_{n-1}, \dots, a_1), \quad \vec{b} := (b_{n-1}, \dots, b_1) \quad (1.62)$$

to

$$\vec{\lambda} := (\lambda_1, \dots, \lambda_n), \quad \vec{q} := (q_1, \dots, q_n) \quad (1.63)$$

can be calculated using the method of wedge products. However, one must first establish some auxiliary results.

Proposition 1.12 *Let $(\mathbf{X})_{11}$ denote the top left hand entry of the matrix \mathbf{X} . We have*

$$((\mathbf{T} - \lambda \mathbf{1})^{-1})_{11} = \sum_{j=1}^n \frac{q_j^2}{\lambda_j - \lambda}. \quad (1.64)$$

Also

$$\prod_{1 \leq i < j \leq n} (\lambda_i - \lambda_j)^2 = \frac{\prod_{i=1}^{n-1} b_i^{2i}}{\prod_{i=1}^n q_i^2}. \quad (1.65)$$

Proof Now

$$((\mathbf{T} - \lambda \mathbf{1})^{-1})_{11} = \vec{e}_1 \cdot (\mathbf{T} - \lambda \mathbf{1})^{-1} \vec{e}_1$$

where $\vec{e}_1 := (1, 0, \dots, 0)^T$. Noting

$$\vec{e}_1 = \sum_{j=1}^n (\vec{e}_1 \cdot \vec{v}_j) \vec{v}_j = \sum_{j=1}^n q_j \vec{v}_j$$

and substituting into the above equation gives (1.64).

To derive (1.65) we begin by recalling that in general for \mathbf{X} a $n \times n$ nonsingular matrix,

$$(\mathbf{X}^{-1})_{11} = \frac{\det \mathbf{X}_{n-1}}{\det \mathbf{X}} \quad (1.66)$$

where \mathbf{X}_{n-1} denotes the bottom right $n-1 \times n-1$ submatrix of \mathbf{X} . Hence we can rewrite (1.64) to read

$$\frac{\prod_{i=1}^{n-1} (\lambda - \lambda_i^{(n-1)})}{\prod_{i=1}^n (\lambda - \lambda_i)} = \sum_{j=1}^n \frac{q_j^2}{\lambda - \lambda_j} \quad (1.67)$$

where $\{\lambda_i^{(n-1)}\}$ denotes the eigenvalues of \mathbf{X}_{n-1} . It follows from this that

$$q_j^2 = \frac{P_{n-1}(\lambda_j)}{P'_n(\lambda_j)}, \quad P_k(\lambda) := \prod_{i=1}^k (\lambda - \lambda_i^{(k)}),$$

where $P_k(\lambda)$ is the characteristic polynomial of the bottom right $k \times k$ submatrix of \mathbf{T} , \mathbf{T}_k say, and $\{\lambda_i^{(k)}\}$ the corresponding eigenvalues. Hence

$$\prod_{i=1}^n q_i^2 = \frac{\prod_{i=1}^n |P_{n-1}(\lambda_i)|}{\prod_{1 \leq i < j \leq n} (\lambda_i - \lambda_j)^2}. \quad (1.68)$$

Next, by expanding along the first row of $\lambda \mathbf{1}_k - \mathbf{T}_k$, one obtains the three term recurrence

$$P_k(\lambda) = (\lambda - a_k)P_{k-1}(\lambda) - b_{k-1}^2 P_{k-2}(\lambda) \quad (1.69)$$

and it follows from this that

$$\prod_{i=1}^{k-1} |P_k(\lambda_i^{(k-1)})| = b_{k-1}^{2(k-1)} \prod_{i=1}^{k-1} |P_{k-2}(\lambda_i^{(k-1)})|.$$

Since

$$\prod_{i=1}^{k-1} |P_{k-2}(\lambda_i^{(k-1)})| = \prod_{i=1}^{k-1} \prod_{j=1}^{k-2} |\lambda_i^{(k-1)} - \lambda_j^{(k-2)}| = \prod_{j=1}^{k-2} |P_{k-1}(\lambda_j^{(k-2)})| \quad (1.70)$$

this can be rewritten as

$$\prod_{i=1}^{k-1} |P_k(\lambda_i^{(k-1)})| = b_{k-1}^{2(k-1)} \prod_{j=1}^{k-2} |P_{k-1}(\lambda_j^{(k-2)})|$$

and iterated to give

$$\prod_{i=1}^{n-1} |P_n(\lambda_i^{(n-1)})| = \prod_{i=1}^{n-1} b_i^{2i}.$$

Further use of (1.70) and substitution into (1.68) gives (1.65).

Proposition 1.13 *The Jacobian for the change of variables (1.62) to (1.63) can be written as*

$$\frac{1}{q_n} \frac{\prod_{i=1}^{n-1} b_i}{\prod_{i=1}^n q_i}. \quad (1.71)$$

Proof Rewriting (1.64) in the form

$$((\mathbf{1} - \lambda \mathbf{T})^{-1})_{11} = \sum_{j=1}^n \frac{q_j^2}{1 - \lambda \lambda_j}$$

and equating successive powers of λ on both sides gives

$$\begin{aligned} 1 &= \sum_{j=1}^n q_j^2, & a_n &= \sum_{j=1}^n q_j^2 \lambda_j, & * + b_{n-1}^2 &= \sum_{j=1}^n q_j^2 \lambda_j^2 \\ * + a_{n-1} b_{n-1}^2 &= \sum_{j=1}^n q_j^2 \lambda_j^3, & * + b_{n-2}^2 b_{n-1}^2 &= \sum_{j=1}^n q_j^2 \lambda_j^4, \\ * + a_{n-2} b_{n-2}^2 b_{n-1}^2 &= \sum_{j=1}^n q_j^2 \lambda_j^5, \dots, & * + a_1 b_1^2 \cdots b_{n-2}^2 b_{n-1}^2 &= \sum_{j=1}^n q_j^2 \lambda_j^{2n+1} \end{aligned}$$

where the * denotes terms involving variables already having appeared on the l.h.s. of preceding equations (thus the variables $a_n, b_{n-1}, a_{n-1}, b_{n-2}, \dots$ occur in a triangular structure). The first of these equations implies

$$q_n dq_n = - \sum_{j=1}^{n-1} q_j dq_j.$$

Taking differentials of the remaining equations, substituting for $q_n dq_n$, and then taking wedge products of both sides (making use of the triangular structure on the l.h.s.) shows

$$\prod_{j=1}^{n-1} b_j^{4j-3} da db = q_n^2 \prod_{j=1}^{n-1} q_j^3 \det \left[[\lambda_k^j - \lambda_n^j]_{\substack{j=1, \dots, 2n-1 \\ k=1, \dots, n-1}} [j \lambda_k^{j-1}]_{\substack{j=1, \dots, 2n-1 \\ k=1, \dots, n}} \right] d\lambda dq$$

where

$$da := \bigwedge_{j=1}^n da_j, \quad db := \bigwedge_{j=1}^{n-1} db_j, \quad d\lambda := \bigwedge_{j=1}^n d\lambda_j, \quad dq := \bigwedge_{j=1}^{n-1} dq_j.$$

The determinant can be evaluated (see Exercises 1.5 q.2) thus showing the Jacobian J can be written

$$J = \frac{1}{q_n} \frac{\prod_{j=1}^{n-1} b_j}{\prod_{j=1}^n q_j} \left(\frac{\prod_{j=1}^n q_j^2}{\prod_{j=1}^{n-1} b_j^{2j-1}} \right)^2 \prod_{1 \leq j < k \leq N} (\lambda_k - \lambda_j)^4.$$

Recalling (1.65) shows J is equal to (1.71).

Using Proposition 1.13, the fact that the tridiagonal matrix of Proposition 1.11 has the same eigenvalue p.d.f. as GOE matrices can be directly verified. Moreover, one can prescribe a tridiagonal matrix with eigenvalue p.d.f. (1.47) for general $\beta > 0$ [50].

Proposition 1.14 *In the notation of Proposition 1.11 define the symmetric tridiagonal matrix*

$$\mathbf{T}_\beta := \begin{bmatrix} \mathbf{N}[0, 1] & \tilde{\chi}_{(N-1)\beta} & & & & \\ \tilde{\chi}_{(N-1)\beta} & \mathbf{N}[0, 1] & \tilde{\chi}_{(N-2)\beta} & & & \\ & \tilde{\chi}_{(N-2)\beta} & \mathbf{N}[0, 1] & \tilde{\chi}_{(N-3)\beta} & & \\ & & \ddots & \ddots & \ddots & \\ & & & \tilde{\chi}_{2\beta} & \mathbf{N}[0, 1] & \tilde{\chi}_\beta \\ & & & & \tilde{\chi}_\beta & \mathbf{N}[0, 1] \end{bmatrix} \quad (1.72)$$

The eigenvalues and first component of the eigenvectors (which form the vector \vec{q}) are independent, with the distribution of the former given by

$$\frac{1}{\tilde{G}_{\beta N}} \prod_{l=1}^N e^{-\lambda_l^2/2} \prod_{1 \leq j < k \leq N} |\lambda_k - \lambda_j|^\beta d\lambda, \quad \tilde{G}_{\beta N} = (2\pi)^{N/2} \prod_{j=0}^{N-1} \frac{\Gamma(1 + (j+1)\beta/2)}{\Gamma(1 + \beta/2)}, \quad (1.73)$$

and the distribution of the latter given by

$$\frac{1}{c_{\beta N} q_N} \prod_{i=1}^N q_i^{\beta-1} dq, \quad q_i > 0, \quad \sum_{i=1}^N q_i^2 = 1, \quad \text{where } c_{\beta N} = \frac{\Gamma^N(\beta/2)}{2^{N-1} \Gamma(\beta N/2)}. \quad (1.74)$$

Proof Denote the joint distribution of \mathbf{T}_β by $P(\mathbf{T}_\beta)$. We have

$$\begin{aligned} P(\mathbf{T}_\beta)(d\mathbf{T}_\beta) &= \frac{2^{N-1}}{(2\pi)^{N/2}} \prod_{l=1}^{N-1} \frac{1}{\Gamma(\beta l/2)} \prod_{l=1}^N e^{-a_l^2/2} b_l^{\beta l-1} e^{-b_l^2} da db \\ &= \frac{2^{N-1}}{(2\pi)^{N/2}} \prod_{l=1}^{N-1} \frac{1}{\Gamma(\beta l/2)} \frac{1}{q_N} \frac{\prod_{l=1}^{N-1} b_l^{\beta l}}{\prod_{l=1}^N q_l} e^{-\text{Tr}(\mathbf{T}_\beta^2)/2} d\lambda dq. \end{aligned}$$

But

$$e^{-\text{Tr}(\mathbf{T}_\beta^2)/2} = e^{-\sum_{j=1}^N \lambda_j^2/2}, \quad \prod_{l=1}^{N-1} b_l^{\beta l} = \prod_{i=1}^N q_i^\beta \prod_{1 \leq i < j \leq N} |\lambda_j - \lambda_i|^\beta$$

where the latter formula follows from (1.65), so indeed the dependence on λ and q factorizes into the functional forms specified in (1.73) and (1.74). The normalization for (1.74) follows from the Dirichlet integral [211] (see also Section 3.5.1 q.3)

$$\int_{\sum_{i=1}^{n+1} \rho_i = 1, \rho_i > 0} d\rho_1 \cdots d\rho_n \prod_{i=1}^{n+1} \rho_i^{s_i-1} = \frac{\Gamma(s_1) \cdots \Gamma(s_{n+1})}{\Gamma(s_1 + \cdots + s_{n+1})} \quad (1.75)$$

with $n = N - 1$, $s_i = \beta$ and the change of variables $\rho_i = q_i^{1/2}$. With this normalization specified, the value of $\tilde{G}_{\beta N}$ follows.

We remark that the evaluation of $\tilde{G}_{\beta N}$ given in (1.73) implies, after a simple change of variables, that the normalization constant in (1.47) has the evaluation

$$G_{\beta N} = \beta^{-N/2 - N\beta(N-1)/4} (2\pi)^{N/2} \prod_{j=0}^{N-1} \frac{\Gamma(1 + (j+1)\beta/2)}{\Gamma(1 + \beta/2)}. \quad (1.76)$$

Another point of interest is that the recurrence (1.69) with

$$a_k \in \mathbb{N}[0, 1], \quad b_k^2 \in \Gamma[k\beta/2, 1] \quad (1.77)$$

can be used to generate the characteristic polynomial for a member of the Gaussian β -ensemble, so the p.d.f. (1.73) can be sampled by simply computing the zeros of this polynomial.

Exercises 1.5

1. The objective of this exercise is to derive the Vandermonde determinant evaluation

$$\det[x_j^{k-1}]_{j,k=1,\dots,N} := \begin{vmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{N-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{N-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_N & x_N^2 & \cdots & x_N^{N-1} \end{vmatrix} = \prod_{1 \leq j < k \leq N} (x_k - x_j). \quad (1.78)$$

(i) Verify that both the determinant and product of differences are antisymmetric polynomials which are homogeneous of degree $\frac{1}{2}N(N-1)$ and hence must be proportional.

(ii) Show that the proportionality constant is unity by comparing the coefficients of the term $x_1^0 x_2^1 \cdots x_N^{N-1}$ on both sides.

2. In this exercise it will be shown that

$$\det \left[[\lambda_k^j - \lambda_n^j]_{\substack{j=1,\dots,2n-1 \\ k=1,\dots,n-1}} [j\lambda_k^{j-1}]_{\substack{j=1,\dots,2n-1 \\ k=1,\dots,n}} \right] = \prod_{1 \leq j < k \leq N} (\lambda_k - \lambda_j)^4.$$

(i) Show that both sides are homogeneous symmetric polynomials of degree $2N(N-1)$, and that the determinant and its first three derivatives with respect to λ_1 vanish at $\lambda_1 = \lambda_2$. Conclude from this that both sides must be proportional.

(ii) Show that the proportionality constant must be unity by comparing coefficients of $(\lambda_1^0 \lambda_2^1 \cdots \lambda_N^{N-1})^4$ on both sides.

1.6 Coulomb gas analogy

The eigenvalue p.d.f. of (1.73) is, up to a simple scale factor, identical to the eigenvalue p.d.f. (1.47) which for $\beta = 1, 2$ and 4 is realized by the GOE, GUE and GSE respectively. Comparison of (1.47) with (1.6) shows immediately that the eigenvalue p.d.f. is identical to the Boltzmann factor of a one-component log-potential Coulomb system confined to a line, with the position of the charged particles corresponding identically to the location of the eigenvalues. Furthermore, the background charge density $-q\rho_b(y)$ is such that

$$\frac{x^2}{2} + C = \int_{-\infty}^{\infty} \rho_b(y) \log|x-y| dy, \quad (1.79)$$

where C is a constant.

Note that it is not possible to satisfy (1.79) for $|x| \rightarrow \infty$, since in this limit the r.h.s. is to leading order $N \log|x|$ and is thus a different order to the l.h.s. Instead we seek to solve the integral equation for $\rho_b(y)$ with support on the finite interval $(-a, a)$ say, and x confined to the same interval. Then (1.79) reads

$$\frac{x^2}{2} + C = \int_{-a}^a \rho_b(y) \log|x-y| dy, \quad x \in (-a, a). \quad (1.80)$$

The solution of the equation can be computed exactly by the method of eigenfunction expansions (see e.g. Porter and Stirling [176]).

Proposition 1.15 *Suppose all the eigenvalues $\{\lambda_n\}_{n=0,1,\dots}$ and corresponding normalized eigenfunctions $\{\phi_n\}_{n=0,1,\dots}$ of a linear operator A are known, all the eigenvalues are nonzero, and that the eigenfunctions form a complete set. Then the operator equation $g = Af$, where g is given, has the solution*

$$f = \sum_{n=0}^{\infty} \frac{\langle g|\phi_n\rangle}{\lambda_n} \phi_n,$$

where $\langle | \rangle$ denotes the inner product.

Proof Since the eigenfunctions form a complete set, $g = \sum_{n=0}^{\infty} \langle g|\phi_n\rangle \phi_n$. Also $f = \sum_{n=0}^{\infty} \langle f|\phi_n\rangle \phi_n$ and so $Af = \sum_{n=0}^{\infty} \langle f|\phi_n\rangle \lambda_n \phi_n$. The result follows by equating the coefficients of ϕ_n in the operator equation.

To make use of this method, it is necessary to make the further change of variables

$$y = a \cos \theta, \quad x = a \cos \sigma, \quad \sin \theta \rho_b(a \cos \theta) =: a\phi(\cos \theta)$$

so that (1.80) reads

$$-\frac{1}{4} \cos 2\sigma - \left(\frac{1}{4} - \frac{N}{a^2} \log a + \frac{C}{a^2} \right) = - \int_0^\pi \log|\cos \theta - \cos \sigma| \phi(\theta) d\theta. \quad (1.81)$$

Note that $\cos \theta - \cos \sigma = 2 \sin(\sigma - \theta)/2 \sin(\sigma + \theta)/2$. Since $2|\sin(\sigma - \theta)/2|$ gives the chord length for two points on the unit circle with angles σ and θ , the r.h.s. of (1.81) can be interpreted as giving (up to an additive constant) the electrostatic potential at the angle σ due to a charge density $\phi(\theta)$ and $\phi(2\pi - \theta)$ between 0 and π and π and 2π respectively on the unit circle.

The eigenvalues and eigenfunctions of the integral operator

$$A[\phi](\sigma) := - \int_0^\pi \log |\cos \theta - \cos \sigma| \phi(\theta) d\theta \quad (1.82)$$

are known (see Exercises 1.6 q.1). They are

$$\lambda_0 = \pi \log 2, \quad \phi_0(\theta) = \frac{1}{\pi^{1/2}}, \quad \lambda_n = \frac{\pi}{n}, \quad \phi_n(\theta) = \left(\frac{2}{\pi}\right)^{1/2} \cos n\theta \quad (n = 1, 2, \dots).$$

In terms of these eigenfunctions

$$-\frac{1}{4} \cos 2\sigma - \left(\frac{1}{4} - \frac{N}{a^2} \log a + \frac{C}{a^2}\right) = -\frac{1}{4} \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \phi_2(\sigma) - \left(\frac{1}{4} - \frac{N}{a^2} \log a + \frac{C}{a^2}\right) \pi^{\frac{1}{2}} \phi_0(\sigma),$$

and so from the general formula of Proposition 1.9 the solution of the transformed integral equation is

$$\phi(\theta) = -\frac{1}{2\pi} (\cos 2\theta - 1) - \frac{1}{\pi \log 2} \left(\frac{1}{4} - \frac{N}{a^2} \log a + \frac{C}{a^2} + \frac{1}{2} \log 2\right),$$

where $1/2\pi$ has been added and subtracted for latter convenience. Reverting back to the original variables we obtain the following result.

Proposition 1.16 *The solution of the integral equation*

$$\frac{x^2}{2} + C = \int_{-a}^a \rho_b(y) \log |x - y| dy, \quad -a \leq x \leq a$$

is

$$\rho_b(y) = \frac{a}{\pi} \sqrt{1 - \left(\frac{y}{a}\right)^2} - \frac{1}{\pi \log 2} \left(\frac{1}{4} - \frac{N}{a^2} \log a + \frac{C}{a^2} + \frac{1}{2} \log 2\right) \frac{a}{\sqrt{1 - (y/a)^2}}.$$

We see that there are two drastically different classes of solution depending on the value of C . Unless we choose

$$C = N \log a - \frac{a^2}{4} - \frac{a^2}{2} \log 2 \quad (1.83)$$

the density profile $\rho_b(y)$ has an inverse square root singularity at $y = \pm a$. However, choosing C according to (1.83) the term proportional to $(1 - (y/a)^2)^{-1/2}$ vanishes and a physically sensible result is obtained. Making this choice of C and fixing a by the neutrality condition $\int_{-a}^a \rho_b(y) dy = N$ gives the desired analogy between the Boltzmann factor of a one-component log-potential Coulomb system and the eigenvalue p.d.f.'s of Proposition 1.8.

Proposition 1.17 *The Boltzmann factor of the one-component log-potential Coulomb system with particles of charge $q = 1$ at x_1, \dots, x_N , confined to the interval $[-\sqrt{2N}, \sqrt{2N}]$ with a neutralizing background charge density*

$$-\rho_b(y) = -\frac{\sqrt{2N}}{\pi} \sqrt{1 - \frac{y^2}{2N}},$$

is

$$A \exp \left(-\frac{\beta}{2} \sum_{j=1}^N x_j^2 \right) \prod_{1 \leq j < k \leq N} |x_k - x_j|^\beta, \quad A = \exp \left(-\frac{\beta N^2}{4} \log(N/2) + \frac{3\beta N^2}{8} \right).$$

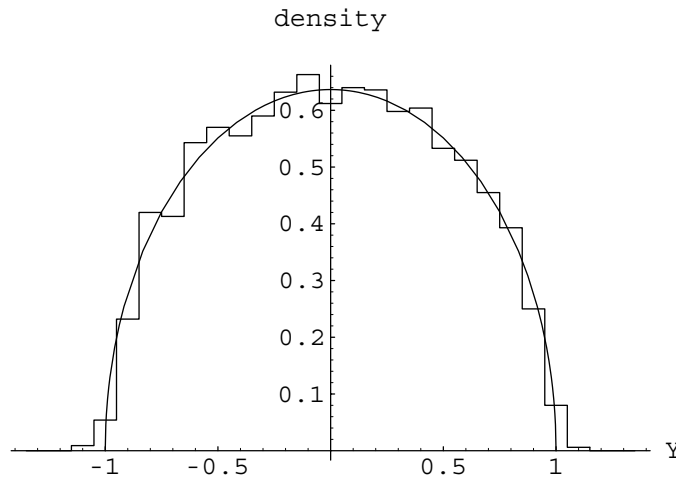


Figure 1.1: Empirical demonstration of the Wigner semi-circle law for 10×10 matrices from the GUE

Proof In the general formula (1.6) for the Boltzmann factor of a one-component log-potential Coulomb system, $r_k = x_k$ and from Proposition 1.16 and (1.83) with $a = \sqrt{2N}$

$$V(x) = \frac{x^2}{2} + 4N \left(\frac{1}{4} \log \sqrt{N/2} - \frac{1}{8} \right)$$

and

$$U_3 = -q^2 \left[\frac{N^2}{\pi} \int_{-1}^1 x^2 \sqrt{1-x^2} dx + 2N^2 \left(\frac{1}{4} \log \sqrt{N/2} - \frac{1}{8} \right) \right].$$

A simple change of variables $x = \cos \theta$ shows that the integral in the above equals $\pi/8$. The stated formula for the Boltzmann factor follows.

Proposition 1.17 can be used to predict the eigenvalue density profile for Gaussian random matrices. Physically, we expect that to leading order in N Coulomb systems are locally charge neutral, which for a one-component system implies that to leading order the particle density will equal the background density. (This statement is put on a more mathematical footing in Exercises ?? q.4 below.) For the log-potential system of Proposition 1.17 this gives the particle density as

$$\rho(y) = \frac{\sqrt{2N}}{\pi} \sqrt{1 - \frac{y^2}{2N}}. \quad (1.84)$$

But the statistical properties of the log-potential system in Proposition 1.17 are identical to the statistical properties of the eigenvalues of the Gaussian random matrices, so we expect that the eigenvalue density profile will to leading order in N be given by this formula, which is known as the *Wigner semi-circle law*. An alternative approach to this result, which avoids physical reasoning, is given in Exercises 1.7 q.1.

In Figure 1.1 we have plotted the empirical eigenvalue density for 1000 10×10 matrices from the GUE, using the variable $Y = y/\sqrt{2N}$. The accuracy of the Wigner semi-circle law is evident.

1.6.1 The complex electric field and calculation of $\rho_b(y)$

The integral equation (1.80) is the special case $V(x) = x^2/2$ of the integral equation

$$V(x) + C = \int_{-a}^a \rho_b(y) \log|x-y| dy, \quad x \in (-a, a). \quad (1.85)$$

We seek the solution such that $\rho_b(y)$ is bounded at $y = \pm a$ and normalized so that

$$\int_{-a}^a \rho_b(y) dy = N. \quad (1.86)$$

If the primary concern is the calculation of $\rho_b(y)$ and not C , an alternative to the method of eigenfunctions used above is to introduce the complex electric field

$$E(z) := - \int_{-a}^a \frac{\rho_b(y)}{z-y} dy. \quad (1.87)$$

Note that for $z \notin [-a, a]$, $E(z)$ is analytic and has the asymptotic behaviour

$$E(z) \underset{|z| \rightarrow \infty}{\sim} -\frac{N}{z}, \quad (1.88)$$

while for $z \sim \pm a$, by the assumption that $\rho_b(y)$ is bounded,

$$E(z) \underset{z \rightarrow \pm a}{\sim} O(\log(z \mp a)). \quad (1.89)$$

Furthermore, it follows by Cauchy's theorem that

$$E^+(x) - E^-(x) = 2\pi i \rho_b(x), \quad x \in (-a, a), \quad (1.90)$$

where

$$E^\pm(x) = \lim_{\epsilon \rightarrow 0^+} E(x \pm i\epsilon),$$

while (1.85) gives

$$\operatorname{Re} E(x) = \frac{1}{2}(E^+(x) + E^-(x)) = -V'(x), \quad \text{for } x \in (-a, a). \quad (1.91)$$

The properties (1.88) through to (1.91) can be used to characterize $E(z)$. For the quantity $W(z) := e^{E(z)}$ these properties specify a scalar *Riemann-Hilbert problem*.

Restricting attention to potentials $V(x)$ even in x , one can check that the function

$$E(z) = -\frac{1}{\pi} \sqrt{z^2 - a^2} \int_{-a}^a \frac{V'(t)}{(z-t)\sqrt{a^2 - t^2}} dt, \quad (1.92)$$

with a such that

$$\frac{1}{\pi} \int_{-a}^a \frac{tV'(t)}{\sqrt{a^2 - t^2}} dt = N \quad (1.93)$$

has the properties (1.88) through to (1.91) and is thus the sought complex electric field. In particular, to verify (1.91), we note that (1.92) gives

$$E^\pm(x) = \mp \frac{i}{\pi} \sqrt{a^2 - x^2} \lim_{\epsilon \rightarrow 0^+} \int_{-a}^a \frac{V'(t)}{(x \pm i\epsilon - t)\sqrt{a^2 - t^2}} dt \quad (1.94)$$

and then make use of Cauchy's theorem. Using this formula in (1.90) gives an explicit formula for $\rho_b(x)$ in terms of the potential V [159].

Proposition 1.18 *In the case $V(x)$ even, the solution of the integral equation (1.85) with $\rho_b(y)$ bounded at $y = \pm a$ and normalized as in (1.86) is*

$$\rho_b(y) = \frac{1}{\pi^2} \sqrt{a^2 - y^2} \int_{-a}^a \frac{V'(y) - V'(t)}{y - t} \frac{1}{\sqrt{a^2 - t^2}} dt, \quad (1.95)$$

where a is specified by (1.93).

Proof After substituting (1.92) in (1.90), we subtract an appropriate multiple of the identity

$$\lim_{\epsilon \rightarrow 0} \left(\int_{-a}^a \frac{1}{(x + i\epsilon - t)\sqrt{a^2 - t^2}} dt - \int_{-a}^a \frac{1}{(x - i\epsilon - t)\sqrt{a^2 - t^2}} dt \right) = 0$$

from both sides to obtain

$$2\pi i \rho_b(x) = -\frac{i}{\pi} \sqrt{a^2 - x^2} \operatorname{Re} \lim_{\epsilon \rightarrow 0^+} \int_{-a}^a \frac{V'(t) - V'(x)}{(x + i\epsilon - t)\sqrt{a^2 - t^2}} dt.$$

The limit can be taken inside the integrand because the numerator vanishes for $x = t$, giving (1.95).

In the special case $V(y) = y^2/2$, (1.95) gives

$$\rho_b(y) = \frac{a}{\pi} \sqrt{1 - (y/a)^2},$$

while it follows from (1.93) that $a = \sqrt{2N}$, in agreement with (1.84). The generalization of (1.95) for $V(x)$ not necessarily even is

$$\rho_b(y) = \frac{1}{\pi^2} \sqrt{(y-a)(b-y)} \int_a^b \frac{V'(y) - V'(t)}{y-t} \frac{1}{\sqrt{(t-a)(b-t)}} dt, \quad (1.96)$$

where a and b are such that

$$\int_a^b \frac{V'(t)}{\sqrt{(t-a)(b-t)}} dt = 0, \quad \frac{1}{\pi} \int_a^b \frac{V'(t)}{\sqrt{(t-a)(b-t)}} dt = N, \quad (1.97)$$

as can be checked from a similar analysis.

It can happen that the solution (1.95) or (1.96) of (1.85) does not in fact correspond to the background density because it becomes negative within the interval $(-a, a)$. An example is the potential $V(x) = -cx^2 + gx^4$ for c large enough. The formula (1.95) gives

$$\rho_b(y) = \frac{1}{\pi} (-2c + 2ga^2 + 4gy^2) \sqrt{a^2 - y^2} \quad (1.98)$$

where, according to (1.95),

$$-ca^2 + \frac{3ga^4}{2} = N. \quad (1.99)$$

The solution (1.98) will take on negative values whenever $c > ga^2$. According to (1.99), for this to happen it is sufficient that $c^2 > 2gN$. In such a circumstance, the original assumption that the support is on a single interval breaks down, and one must seek a solution supported on a double interval $(-a_2, -a_1) \cup (a_1, a_2)$.

Exercises 1.6

1. (i) Assuming the validity of the formula

$$\log|1 - ae^{ix}| = - \sum_{n=1}^{\infty} \frac{a^n \cos nx}{n}, \quad 0 \leq a < 1, \quad x \in \mathbf{R}$$

for $a = 1$ provided $x \neq 0 \pmod{2\pi}$, deduce that

$$\log|2 \sin(x-t)/2| = - \sum_{n=1}^{\infty} \frac{\cos n(x-t)}{n}$$

for $x-t \neq 0 \pmod{2\pi}$ and write down a similar formula for $\log|2 \sin(x+t)/2|$. Hence derive the cosine expansion

$$\log(2|\cos x - \cos t|) = - \sum_{n=1}^{\infty} \frac{2}{n} \cos nx \cos nt. \quad (1.100)$$

(ii) Use the above cosine expansion to verify that the eigenvalues and normalized eigenfunctions of the integral operator

$$A[\hat{\phi}](\sigma) := - \int_0^{\pi} \log|\cos \theta - \cos \sigma| \hat{\phi}(\theta) d\theta$$

are as specified below (1.82).

2. The objective of this exercise is to compute the background density and the Boltzmann factor for the one-component log-potential system confined to the interval $(-a, a)$ with one-body potential

$$V(x) = \frac{x^2}{2} + g \frac{x^4}{N} + C. \quad (1.101)$$

This calculation is of interest in the graphical expansion of matrix integrals [31, 219], and will be used in this context in the next section.

(i) With $Y = \cos \theta$ verify that

$$\frac{\cos 4\theta - 1}{\sin \theta} = -8Y^2(1 - Y^2)^{1/2}.$$

(ii) Use the eigenfunction expansion method, the result of (i) and Proposition 1.16 to show that the solution of the integral equation

$$V(x) = \int_{-a}^a \rho_b(y) \log|x-y| dy, \quad -a \leq x \leq a$$

which is bounded at $y = \pm a$ is

$$\rho_b(y) = \frac{a}{\pi} \left(1 + \frac{2ga^2}{N} + \frac{4g}{N} y^2 \right) \sqrt{1 - (y/a)^2}$$

provided

$$C = -a^2 \left(\frac{1}{4} - \frac{N}{a^2} \log a + \frac{1}{2} \log 2 + \frac{3ga^2}{8N} + \frac{3ga^2}{2N} \log 2 \right).$$

(iii) Use the neutrality condition to show

$$\frac{a^2}{2} + \frac{3ga^4}{2N} = N, \quad (1.102)$$

and use this in the formula for C to obtain the simplification

$$C = -\frac{a^2}{8} + N \log \frac{a}{2} - \frac{N}{4}.$$

(iv) Use the trigonometric Euler integral in Exercises 3.1 q.1(i) below and the neutrality condition to show

$$U_3/q^2 = -\frac{CN}{2} + \frac{a^4}{192} - \frac{a^2N}{24} - \frac{N^2}{16}$$

and thus

$$\begin{aligned} & (U_2 + U_3)/q^2 - (U_2 + U_3)/q^2 \Big|_{g=0} \\ &= \frac{g}{N} \sum_{j=1}^N x_j^4 - \frac{N^2}{2} \left[\frac{1}{24} ((a/\sqrt{2N})^2 - 1)(9 - (a/\sqrt{2N})^2) - \log(a/\sqrt{2N}) \right], \end{aligned} \quad (1.103)$$

where $(U_2 + U_3)/q^2 \Big|_{g=0}$ is as implicit in Proposition 1.17.

3.(i) For a general potential $u(x)$, use the eigenfunction expansion method to show that the solution $\rho_b(y)$ of the integral equation

$$u(x) + C = \int_{-a}^a \rho_b(y) \log|x-y| dy, \quad x \in [-a, a], \quad (1.104)$$

which is bounded at $y = \pm a$ can be written

$$\rho_b(a \cos \theta) = -\frac{2}{a\pi^2 \sin \theta} \sum_{p=1}^{\infty} p \left(\int_0^\pi u(a \cos \sigma) \cos p\sigma d\sigma \right) (\cos p\theta - 1).$$

(ii) For $u(x) = x^{2n}$, $n \in \mathbf{Z}^+$, use the integration formula

$$\int_0^\pi \cos^{2n} \sigma \cos 2p\sigma d\sigma = \frac{\pi}{2^{2n}} \binom{2n}{n+p},$$

verified using complex exponentials, and the transformation identity

$$\frac{1}{2n} \sum_{p=1}^n p \binom{2n}{n+p} \frac{1 - \cos 2p\theta}{1 - \cos^2 \theta} = \sum_{l=1}^n \binom{2(n-l)}{n-l} (2 \cos \theta)^{2(l-1)}$$

to show that [32]

$$\rho_b(x) = \frac{4n}{\pi} \left(\frac{a}{2}\right)^{2n-1} \left(\sum_{l=1}^n \binom{2(n-l)}{n-l} \left(\frac{2x}{a}\right)^{2(l-1)} \right) \sqrt{1 - \left(\frac{x}{a}\right)^2}.$$

Check that this is consistent with $\rho_b(y)$ in q.2(ii).

4. In this exercise the location of the minimum of the function

$$H(x_1, \dots, x_N) := \frac{1}{2} \sum_{j=1}^N x_j^2 - \sum_{1 \leq j < k \leq N} \log|x_k - x_j|$$

will be determined by following a calculation of Stieltjes [191]. This gives the equilibrium points of the system of Proposition 1.17.

(i) Show that H is convex by establishing that for $t_j \neq 0$ ($j = 1, \dots, N$), $\sum_{j \neq k=1}^N t_j t_k \frac{\partial H}{\partial x_j \partial x_k} > 0$, and conclude that H has a unique minimum.

(ii) Let $g(x) = \prod_{l=1}^N (x - x_l^{(0)})$. Show that the equations for the minimum $\partial H / \partial x_j = 0$ ($j = 1, \dots, N$) can be written

$$g''(x_j) - 2x_j g'(x_j) = 0 \quad (j = 1, \dots, N)$$

(iii) Observe that the l.h.s. of the above equation is a polynomial of degree N which vanishes at the zeros of $g(x)$, and so must be proportional to $g(x)$, to deduce the d.e.

$$g''(x) - 2xg'(x) + 2Ng(x) = 0.$$

Hence show that the minimum of $H(x_1, \dots, x_N)$ occurs at the zeros of the Hermite polynomial $H_N(x)$.

1.7 Matrix integrals and combinatorics

1.7.1 Combinatorics of $\langle \text{Tr}(\mathbf{X}^{2k}) \rangle_{\text{GUE}^*}$

In this section we put our knowledge of the asymptotic density for the GUE to use in the solution of a combinatorial problem. It has long been known [31] that the matrix integrals

$$\int f(\mathbf{X}) e^{-\text{Tr}(\mathbf{X}^2)/2} (d\mathbf{X})$$

for \mathbf{X} a particular class of matrices and suitable $f(\mathbf{X})$ have combinatorial significance in that they count certain diagrams embedded on surfaces according to their genus. Here, following [221, 84], we will detail such a combinatorial interpretation of the matrix integral

$$\int \text{Tr}(\mathbf{X}^{2k}) e^{-\text{Tr}(\mathbf{X}^2)/2} (d\mathbf{X}) =: \langle \text{Tr}(\mathbf{X}^{2k}) \rangle_{\text{GUE}^*}, \quad (1.105)$$

where it is assumed $(d\mathbf{X})$ is normalized, and GUE^* is identical to the GUE except that $\mathbf{X} \mapsto \mathbf{X}/\sqrt{2}$. By changing variables $\mathbf{X} = \mathbf{U}\mathbf{L}\mathbf{U}^{-1}$ to the eigenvalues and eigenvectors we see from the result of Exercises 1.4 q.3 that

$$\langle \text{Tr}(\mathbf{X}^{2k}) \rangle_{\text{GUE}^*} = \frac{1}{C} \int_{-\infty}^{\infty} d\lambda_1 \cdots \int_{-\infty}^{\infty} d\lambda_N \prod_{l=1}^N e^{-\lambda_l^2/2} \left(\sum_{j=1}^N \lambda_j^{2k} \right) \prod_{1 \leq j < k \leq N} |\lambda_k - \lambda_j|^2. \quad (1.106)$$

Changing variables $\lambda_l \mapsto \sqrt{2}\lambda_l$ we see from the definition (1.15) that in terms of the density $\rho_{(1)}(\lambda)$ for the GUE we have

$$\langle \text{Tr}(\mathbf{X}^{2k}) \rangle_{\text{GUE}^*} = 2^k \int_{-\infty}^{\infty} \lambda^{2k} \rho_{(1)}(\lambda) d\lambda. \quad (1.107)$$

However it is not from (1.106) that the combinatorics arise; this comes from the evaluation of (1.105) as a Gaussian integral over the independent elements of the matrix \mathbf{X} . The latter task can be achieved by using a particular matrix version of *Wick's theorem*.

Proposition 1.19 *Let $\mathbf{X} = [z_{jk}]_{j,k=1,\dots,N}$, $z_{jk} = x_{jk} + iy_{jk}$ be Hermitian so that*

$$e^{-\text{Tr}(\mathbf{X}^2)/2} (d\mathbf{X}) = \frac{1}{C} e^{-\text{Tr}(\mathbf{X}^2)/2} \prod_{j=1}^N dx_{jj} \prod_{1 \leq j < k \leq N} dx_{jk} dy_{jk}.$$

Let I be a finite ordered set of pairs of indices (j, k) , $1 \leq j, k \leq N$, and let P denote a matching of the elements of I in pairs. Then we have

$$\left\langle \prod_{(i,j) \in I} z_{ij} \right\rangle_{\text{GUE}^*} = \sum_{\substack{\text{pairings} \\ P \text{ of } I}} \prod_{(i,j), (k,l)} \langle z_{ij} z_{kl} \rangle_{\text{GUE}^*}. \quad (1.108)$$

Proof Introducing the Hermitian matrix $\mathbf{Y} = [w_{jk}]_{j,k=1,\dots,N}$ we observe that

$$\left\langle \prod_{(i,j) \in I} z_{ij} \right\rangle_{\text{GUE}^*} = \left(\prod_{(i,j) \in I} \frac{\partial}{\partial w_{ji}} \right) \left\langle e^{\text{Tr}(\mathbf{Y}\mathbf{X})} \right\rangle_{\text{GUE}^*} \Big|_{\mathbf{Y}=\mathbf{0}}. \quad (1.109)$$

But since

$$\mathrm{Tr} \mathbf{X}^2 - 2\mathrm{Tr}(\mathbf{Y} \mathbf{X}) = \mathrm{Tr}((\mathbf{X} - \mathbf{Y})^2) - \mathrm{Tr} \mathbf{Y}^2$$

we see from the change of variables $\mathbf{X} \mapsto \mathbf{X} + \mathbf{Y}$ that

$$\left\langle e^{\mathrm{Tr}(\mathbf{Y} \mathbf{X})} \right\rangle_{\mathrm{GUE}^*} = e^{\mathrm{Tr}(\mathbf{Y}^2)/2} = \prod_{j=1}^N e^{w_j^2/2} \prod_{1 \leq j < k \leq N} e^{w_j w_k}.$$

Thus (1.109) gives

$$\left\langle \prod_{(i,j) \in I} z_{ij} \right\rangle_{\mathrm{GUE}^*} = \sum_{\substack{\text{pairings } P \\ \text{of } I}} \prod \delta_{i,l} \delta_{j,k}$$

which reduces to (1.108) after noting

$$\langle z_{ij} z_{kl} \rangle_{\mathrm{GUE}^*} = \delta_{i,l} \delta_{j,k}. \quad (1.110)$$

Our task is to compute

$$\langle \mathrm{Tr}(\mathbf{X}^{2k}) \rangle_{\mathrm{GUE}^*} := \left\langle \sum_{i_1, \dots, i_{2k}=1}^N z_{i_1 i_2} z_{i_2 i_3} \cdots z_{i_{2k-1} i_{2k}} z_{i_{2k} i_1} \right\rangle_{\mathrm{GUE}^*}. \quad (1.111)$$

According to (1.108) we have

$$\langle \mathrm{Tr}(\mathbf{X}^{2k}) \rangle_{\mathrm{GUE}^*} = \sum_{i_1, \dots, i_{2k}=1}^N \sum_{\substack{\text{pairings } P \text{ of} \\ \{(i_1, i_2), (i_2, i_3), \dots, (i_{2k}, i_1)\}}} \prod_{(j,j'), (l,l')} \langle z_{ij} z_{i'j'} \rangle_{\mathrm{GUE}^*} \quad (1.112)$$

and (1.110) shows that various labels must coincide for a given term in this expression to be non-zero. For example, with $k = 4$ consider the particular term in (1.112)

$$\langle z_{i_1 i_2} z_{i_3 i_4} \rangle \langle z_{i_2 i_3} z_{i_8 i_1} \rangle \langle z_{i_4 i_5} z_{i_6 i_7} \rangle \langle z_{i_5 i_6} z_{i_7 i_8} \rangle = (\delta_{i_1, i_4} \delta_{i_2, i_3}) (\delta_{i_2, i_1} \delta_{i_3, i_8}) (\delta_{i_4, i_7} \delta_{i_5, i_6}) (\delta_{i_5, i_8} \delta_{i_6, i_7}). \quad (1.113)$$

For this to be non-zero we must have $i_1 = i_2 = \dots = i_8$, giving only one independent label. As another example, consider the term

$$\langle z_{i_1 i_2} z_{i_4 i_5} \rangle \langle z_{i_2 i_3} z_{i_3 i_4} \rangle \langle z_{i_5 i_6} z_{i_8 i_1} \rangle \langle z_{i_6 i_7} z_{i_7 i_8} \rangle = (\delta_{i_1 i_5} \delta_{i_2 i_4}) (\delta_{i_2 i_4}) (\delta_{i_5 i_1} \delta_{i_6 i_8}) (\delta_{i_6 i_8}) \quad (1.114)$$

which is non-zero for $i_1 = i_5$, $i_2 = i_4$, $i_6 = i_8$ giving five independent labels i_1, i_2, i_3, i_6, i_7 say.

In general the non-zero terms in (1.112) can be represented graphically in two related ways, both of which involve a regular $2k$ -gon, with the vertices labelled i_1, \dots, i_{2k} , and the edges orientated clockwise. One method to carry out the pairing between consecutive vertices (i_j, i_k) and consecutive vertices (i_l, i_m) is to join the corresponding edges on the $2k$ -gon according to the rule that edges must be joined in opposite directions (see Figure 1.2 for this representation of (1.113) and (1.114)).

Another approach to carry out the pairing is to draw a straight line segment perpendicular to and outwards from the ends of each edge of the $2k$ -gon. These segments are to be given the directions of out and in alternately around the $2k$ -gon. Then each non-zero contribution to (1.112) can be represented by joining different pairs (i_j, i_k) and (i_l, i_m) of parallel straight line

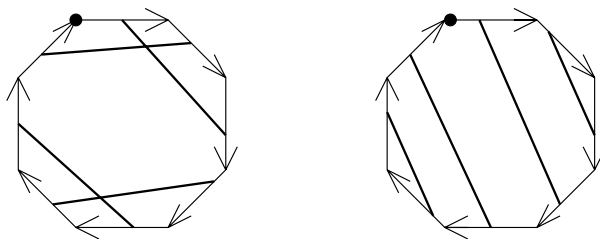


Figure 1.2: Graphical representation of the contributions (1.113) and (1.114) respectively. The heavy lines identify edges and the dot marks the location of the vertex labelled i_1 , with the other vertices labelled clockwise.

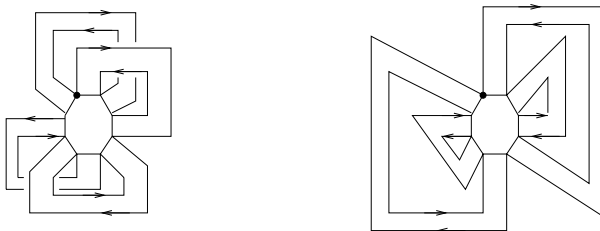


Figure 1.3: The dual graphical representation of Figure 1.2 for the contributions of (1.113) and (1.114) respectively. The dot marks the location of the vertex labelled i_1 , with the other vertices labelled clockwise.

segments to form roadways (see Figure 1.3 for this representation of (1.113) and (1.114)). Note that the joining is such that edges of the roadways have definite directions.

The diagrams of Figure 1.2 can be catalogued according to the number ν of independent vertices after pairing. This of course is just the number of independent summation labels in (1.112) so we can write

$$\langle \text{Tr}(\mathbf{X}^{2k}) \rangle_{\text{GUE}^*} = \sum_{\nu=1}^{k+1} a_{\nu}(k) N^{\nu} \quad (1.115)$$

where $a_{\nu}(k)$ denotes the number of different pairings which have ν vertices. On the other hand, the diagrams of Figure 1.3 are topological duals of Figure 1.2 with the ν independent vertices of the latter now ν independent faces. These can be determined by following the edge of a roadway and its continuation according to its direction, until arriving back at the starting point. The formal meaning of the faces is obtained by shrinking the width of the roadways in Figure 1.3 to single lines, at the same time as shrinking the $2k$ -gon to a single vertex so the lines become loops, then embedding the diagram on a closed surface as a *map*.

Definition 1.9 *A map is a graph (i.e. collection of vertices and edges) drawn on a closed surface such that the edges do not intersect and, if we cut the surface along the edges, a disjoint union of sets topologically equivalent to an open disk results. The number of such disks is by definition the number of faces of the map.*

It is similarly the case that the number of independent vertices in the diagrams of Figure 1.2 can be specified in terms of the corresponding map.

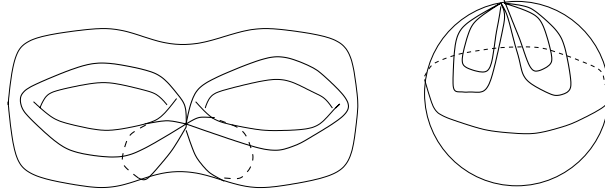


Figure 1.4: Embedding of the graphs of Figure 1.3 (after shrinking the k -gon to a single vertex, and the roadways to single lines) onto a closed surface to form a map with a single vertex.

The index ν in (1.115) determines the genus g (number of holes) of the closed surface. This follows from *Euler's relation*

$$2 - 2g = V - E + F \quad (1.116)$$

where V denotes the number of vertices, E the number of edges and F the number of faces. In the diagrams of Figure 1.2 $V = \nu$, $F = 1$ and $E = k$, while in the diagrams of Figure 1.3 the role of V and F is interchanged so that $V = 1$, $F = \nu$ and $E = k$. Either way (1.116) gives

$$\nu = k + 1 - 2g. \quad (1.117)$$

As shown in Figure 1.4, the diagrams of Figure 1.3 can be directly embedded on a surface, thereby illustrating (1.117) (thus in the first case $k = 4$, $\nu = 1$, $g = 2$ while in the second case $k = 4$, $\nu = 5$, $g = 0$).

Using (1.117) in (1.115) gives

$$\langle \text{Tr}(\mathbf{X}^{2k}) \rangle_{\text{GUE}^*} = N^{k+1} \sum_{g=0}^{\lfloor k/2 \rfloor} a_{k+1-2g}(k) N^{-2g}. \quad (1.118)$$

In particular

$$\lim_{N \rightarrow \infty} N^{-k-1} \langle \text{Tr}(\mathbf{X}^{2k}) \rangle_{\text{GUE}^*} = a_{k+1}(k) \quad (1.119)$$

where $a_{k+1}(k)$ denotes the number of matchings of the $2k$ -gon which are planar (i.e. can be embedded on the surface of a sphere, which has $g = 0$). Substituting for $\langle \text{Tr}(\mathbf{X}^{2k}) \rangle_{\text{GUE}^*}$ using (1.106), and then substituting for $\rho_{(1)}(x)$ using (1.84), evaluating the integral using (3.3) below and simplifying the resulting gamma functions using the duplication formula

$$2^{2z-1} \Gamma(z) \Gamma(z + 1/2) = \pi^{1/2} \Gamma(2z), \quad (1.120)$$

one finds

$$a_{k+1}(k) = \frac{1}{k+1} \binom{2k}{k}. \quad (1.121)$$

This number is familiar in combinatorics and is called the k th *Catalan number*. In fact (1.121) can easily be derived without using (1.107), which then has the significance of providing an alternative derivation of the Wigner semi-circle law for Hermitian matrices (1.84) (see Exercises 1.7 q.1). However this is not the case for the coefficients $a_{k-1}(k)$, $a_{k-3}(k)$, \dots for which the use of (1.107) is the most efficient. We will return to the evaluation of these numbers in Chapter 4 when the exact value of $\rho_{(1)}(\lambda)$ is available.

We remark that $\langle \text{Tr}(\mathbf{X}^{2k}) \rangle_{\text{GOE}}$ allows for a similar combinatorial description in terms of maps on surfaces, although now the surfaces may be nonorientable (corresponding to graphs with twisted ribbons) [134].

1.7.2 Combinatorics of the $\beta = 2$ partition function with a general power series potential

Closely related to the combinatorial interpretation of (1.105) is the combinatorial interpretation of

$$Z_N(\{g_j\}) := \left\langle \prod_{l=1}^N e^{\sum_{j=1}^{\infty} g_j x_l^j / j N^{j/2-1}} \right\rangle_{\text{GUE}^*} \quad (1.122)$$

when expanded in a power series in $\{g_i\}$. For the latter, expanding the exponentials gives

$$Z_N(\{g_j\}) = \sum_{n_1, n_2, \dots = 0}^{\infty} \prod_{j=1}^{\infty} \frac{g_j^{n_j}}{j^{n_j} n_j! N^{j/2-1}} \left\langle \prod_{l=1}^N \left(\sum_{j=1}^{\infty} x_l^j \right)^{n_j} \right\rangle_{\text{GUE}^*}, \quad (1.123)$$

while for the average we have

$$\left\langle \prod_{j=1}^{\infty} \left(\sum_{l=1}^N x_l^j \right)^{n_j} \right\rangle_{\text{GUE}^*} = \left\langle \prod_{j=1}^{\infty} \left(\text{Tr } \mathbf{X}^j \right)^{n_j} \right\rangle_{\text{GUE}^*}. \quad (1.124)$$

From the discussion of the previous subsection we know how to give a combinatorial interpretation of (1.124) in the special case $n_k = 1, n_j = 0$ ($j \neq k$). A natural generalization of this interpretation extends to the general case [221, 84].

Each factor of $\text{Tr } \mathbf{X}^j$ is represented as a j -gon with vertices labelled i_1, i_2, \dots, i_j clockwise, starting at a marked vertex. These labels on vertices extend to labels on pairs of oppositely directed roadway edges coming into and out of each vertex. Whereas in the case of a single factor of $\text{Tr } \mathbf{X}^j$ the combinatorial interpretation of computing (1.124) via Wick's theorem involved connecting roadways within the single j -gon, the graphical representation of contributions to (1.124) is to connect roadways amongst or within any of the n_j j -gons ($j = 1, 2, \dots$). The resulting structure, referred to as a *labelled fatgraph*, has weight N^ν where ν is the number of faces (which in turn is equal to the number of unpaired labels). The number of edges is equal to $\sum_{j=1}^{\infty} j n_j / 2$ which is required to be an integer, while the number of vertices — defined as the number of j -gons — is equal to $\sum_{j=1}^{\infty} n_j$. Recalling Euler's relation (1.116) we see that (1.124) can thus be written

$$Z_N(\{g_j\}) = \sum_{n_1, n_2, \dots = 0}^{\infty} \left(\prod_{j=1}^{\infty} \frac{g_j^{n_j}}{j^{n_j} n_j!} \right) \sum_g a_g(\{n_j\}) N^{2-2g},$$

where $a_g(\{n_j\})$ is the number of labelled graphs constructed out of n_j j -gons ($j = 1, 2, \dots$) which can be embedded on a surface of genus g .

The various j -gons in the labelled fatgraph will not in general be connected. However as $Z_N(\{g_j\})$ is an exponential generating function for these quantities, it is a well known fact that taking the logarithm restricts to connected components (recall Exercises 1.1 q.6). Thus, denoting this restriction by an asterisk, we have

$$\log Z_N(\{g_j\}) = \sum_{n_1, n_2, \dots = 0}^{\infty} \left(\prod_{j=1}^{\infty} \frac{g_j^{n_j}}{j^{n_j} n_j!} \right) \sum_g^* a_g(\{n_j\}) N^{2-2g}. \quad (1.125)$$

Fatgraphs which are topologically equivalent define a class of maps Γ . For each class the maximum value of $a_g(\{n_j\})$ is $\prod_{j=1}^{\infty} j^{n_j} n_j!$ and furthermore $\prod_{j=1}^{\infty} j^{n_j} n_j! / a_g(\{n_j\})$ is an integer written

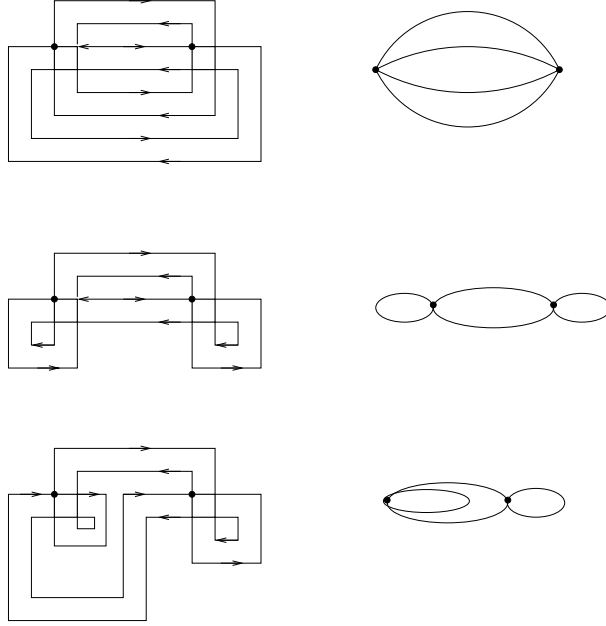


Figure 1.5: There are three classes of fatgraphs which can be constructed out of two 4-gons. An example from each class, together with the corresponding map is given. For the first class $|\text{Aut } \Gamma| = 8$, while for the second and third class $|\text{Aut } \Gamma| = 2$.

as $|\text{Aut } \Gamma|$. As the notation suggests, $|\text{Aut } \Gamma|$ is in fact equal to the order of the group of automorphisms associated with Γ . This can be specified as the number of equivalent labellings of the faces of Γ , which means the number of different labellings in the plane which result from topological transformation of the map on the sphere.

In terms of $|\text{Aut } \Gamma|$ (1.125) reads

$$\log Z_N(\{g_j\}) = \sum_{\text{connected } \Gamma} \frac{1}{|\text{Aut } \Gamma|} N^{2-2g(\Gamma)} \prod_{j=1}^{\infty} g_j^{V_j(\Gamma)}, \quad (1.126)$$

where n_j in (1.125) has been written $V_j(\Gamma)$ in (1.126) to emphasize that it counts the number of vertices with coordination number j in the corresponding map. In particular

$$\lim_{N \rightarrow \infty} \frac{1}{N^2} \log Z_N(\{g_j\}) = \sum_{\substack{\text{connected } \Gamma \\ g(\Gamma)=0}} \frac{1}{|\text{Aut } \Gamma|} \prod_{j=1}^{\infty} g_j^{V_j(\Gamma)}, \quad (1.127)$$

and thus we obtain a generating function for maps weighted according to the coordination number of the vertices. To illustrate (1.127), in Figure 1.5 we display the contributions to the coefficient of g_4^2 in terms of fatgraphs, the corresponding maps and their duals.

Suppose $g_j = 0$ for $j \neq 4$. From the definition (1.122) we have

$$\lim_{N \rightarrow \infty} \frac{1}{N^2} \log Z_N(\{g_j\} \Big|_{g_j=0 (j \neq 4)}) = \lim_{N \rightarrow \infty} \frac{1}{N^2} \log \left\langle \prod_{l=1}^N e^{g_4 x_l^4 / 4N} \right\rangle_{\text{GUE}^*}$$

$$= \lim_{N \rightarrow \infty} \frac{1}{N^2} \log \left\langle \prod_{l=1}^N e^{g_4 x_l^4 / N} \right\rangle_{\text{GUE}}. \quad (1.128)$$

To evaluate this limit we make use of the log-gas interpretation of the average as a ratio of configuration integrals relating to one-component log-potential systems with particular neutralizing background charge densities. The Boltzmann factor for these systems contains constant terms (i.e. terms independent of the particle coordinates) which are not present in (1.128). If these terms, A_N say, were included its logarithm would then be expected to be proportional to N as according to (1.8) the difference between two free energies is being calculated. Thus one expects

$$\lim_{N \rightarrow \infty} \frac{1}{N^2} \log Z_N(\{g_j\} |_{g_j=0 (j \neq 4)}) = \lim_{N \rightarrow \infty} \frac{1}{N^2} \log \frac{1}{A_N}.$$

The value of $1/A_N$ has been calculated in Exercises 1.6 q.2(iv). It is equal to the exponential of the x_j independent terms in (1.103) with g in (1.102) replaced by $-g_4$. This gives

$$\lim_{N \rightarrow \infty} \frac{1}{N^2} \log Z_N(\{g_j\} |_{g_j=0 (j \neq 4)}) = -\left(\frac{1}{24}(u-1)(9-u) - \frac{1}{2} \log u\right) \quad (1.129)$$

where u is defined in terms of g_4 as the solution of

$$u - 3g_4 u^2 = 1, \quad u \rightarrow 1 \text{ as } g_4 \rightarrow 0. \quad (1.130)$$

According to the result of Exercises 1.7 q.1

$$u = -\sum_{k=0}^{\infty} \frac{1}{k+1} \binom{2k}{k} (3g_4)^k \quad (1.131)$$

After substituting this in the r.h.s. of (1.129), and substituting (1.127) in the l.h.s, the following result is obtained [31].

Proposition 1.20 *We have*

$$\sum_{\substack{\text{connected } \Gamma \\ g(\Gamma)=0}} \frac{1}{|\text{Aut } \Gamma|} g_4^{V_4(\Gamma)} = \sum_{k=1}^{\infty} \frac{(2k-1)!}{k!(k+2)!} (3g_4)^k. \quad (1.132)$$

Proof The remaining task is to expand the functions of u on the r.h.s. of (1.129) as power series in g_4 . For the quadratic, this is immediate from (1.131) and (1.130). For $\log u$ this follows from the result of Exercises 1.7 q.3.

Let us denote the coefficient of g_4^k in (1.132) by a_k , which then represents the number of (weighted) planar fatgraphs that can be constructed out of k 4-gons. Making use of Stirling's formula

$$\Gamma(x+1) \sim (2\pi x)^{1/2} e^{x \log x - x} \quad \text{as } x \rightarrow \infty, \text{ Re}(x) > 0, \quad (1.133)$$

shows that

$$a_k \sim \frac{12^k}{k^{7/2} \pi}.$$

The particular value of the exponent of the algebraic term $k^{-7/2}$ has meaning in the conformal field theory associated with the graphical expansion [84].

If we cut an edge in any of the planar maps giving rise to (1.132), we obtain a planar fatgraph constructed from 4-gons, but now with two external legs in the same face. The external legs, when distinguished by different labellings, break the symmetry of the maps, so for all classes Γ_2 of such maps $|\Gamma_2| = 1$. Because the legs have been distinguished, and because there are twice as many edges as vertices, one sees [86]

$$\tilde{G} = 1 + 4g_4 \frac{\partial}{\partial g_4} G$$

where \tilde{G} denotes the generating function for the maps with external legs, and G denotes the l.h.s. of (1.132). Substituting the r.h.s. of (1.132) we see that the power series of \tilde{G} has positive integer coefficients, as it must. In particular, the coefficient of g_4^2 is 9. One contribution results from the first map in Figure 1.5, while four result from each of the other two maps therein.

Exercises 1.7

1. Here the number $c_k := a_{k+1}(k)$ of diagrams which can be constructed from a $2k$ -gon according to the prescription of Figure 1.2, and which contain no intersecting lines, will be computed directly.

(i) Suppose the lines from edge 1 join the lines from edge $2j$ ($j = 1, \dots, k$). Argue that inside these lines there can be c_{j-1} configurations of the allowed type, while there are c_{k-j} configurations of the allowed type joining the edges $2j+1, \dots, 2k$. Hence deduce that

$$c_k = \sum_{j=0}^{k-1} c_j c_{k-1-j}, \quad c_0 = 1. \quad (1.134)$$

(ii) Verify that the Catalan numbers solve this recurrence.

(iii) Introduce the generating function $C(t) = \sum_{k=0}^{\infty} c_k t^k$. Use the recurrence (1.134) to show that $C(t)$ satisfies the quadratic equation

$$C(t) = 1 + t(C(t))^2, \quad (1.135)$$

and consequently has the explicit form

$$tC(t) = \frac{1}{2}(1 - (1 - 4t)^{1/2}). \quad (1.136)$$

(iv) With the value of $a_{k+1}(k)$ known independently of the average in (1.119) according to the result of (ii), use (1.107) to deduce that the scaled density

$$\tilde{\rho}_{(1)}(x) := \lim_{N \rightarrow \infty} \sqrt{2/N} \rho_{(1)}(\sqrt{2N}x) \quad (1.137)$$

is such that

$$2^{2k} \int_{-\infty}^{\infty} x^{2k} \tilde{\rho}_{(1)}(x) dx = \frac{1}{k+1} \binom{2k}{k}$$

while the odd moments vanish.

(v) A sufficient condition for a density function to be determined by its moments $\{c_0, c_1, c_2, \dots\}$ is that

$$\sum_{k=0}^{\infty} \frac{c_k t^k}{k!} \quad (1.138)$$

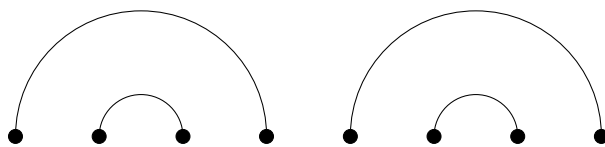


Figure 1.6: The arch configuration equivalent to the second graph in Figure 1.4.

converges for some $t > 0$ (this implies the Fourier transform of the density function is analytic in the neighbourhood of the origin). Verify that this is the case for the moments in (iv). Now use the fact that

$$\tilde{\rho}_{(1)}(x) = \begin{cases} \frac{2}{\pi}(1-x^2)^{1/2}, & |x| < 1 \\ 0, & |x| \geq 1 \end{cases} \quad (1.139)$$

reproduces these values to conclude from (iv) that this is the explicit form of $\tilde{\rho}_{(1)}(x)$.

2. Show that the planar graphs drawn on the $2k$ -gon as illustrated in Figure 1.3 can equivalently be represented as a system of k non-overlapping arches connecting $2k$ points in the line (see Figure 1.6 for an example).

3. [211] Let $f(z)$ and $\phi(z)$ be analytic in a neighbourhood Ω of $z = a$. According to the Lagrange inversion formula, for t small enough that $|t\phi(z)| < |z - a|$, $z \in \Omega$, the equation $\zeta = a + t\phi(\zeta)$ has one solution in Ω , and furthermore

$$f(\zeta) = f(a) + \sum_{n=1}^{\infty} \frac{t^n}{n!} \frac{d^{n-1}}{da^{n-1}} \left(f'(a)(\phi(a))^n \right).$$

Use this formula to show that for x defined as the solution of the equation $x = 1 + yx^p$ with the property $x \rightarrow 1$ as $y \rightarrow 0$, we have

$$\log x = \sum_{k=1}^{\infty} \frac{(kp-1)!}{k!(kp-k)!} y^k.$$

1.8 Convergence

Consider for definiteness GUE matrices. As stated the Wigner semi-circle law tells us the leading large N form of $\langle \frac{1}{N} \sum_{j=1}^N \delta(y - \lambda_j) \rangle_{\text{GUE}}$. As the normalized empirical density integrated over an interval $[a, b]$ is the proportion of eigenvalues in that interval, $\#[a, b]$ say, equivalently the Wigner semi-circle law tells us the expected value of this quantity when averaged over GUE matrices. Indeed this was how Figure 1.1 was produced, with the theoretical means in each bin of the bar graph substituted by their empirical averages.

What if instead one considers $\#[a, b]$ for a sequence of single $n \times n$, $n = 1, 2, \dots$, each chosen from the GUE. Does the resulting sequence of values for $\#[a, b]$ converge to that predicted by the Wigner semi-circle law? And what is the meaning of convergence in this setting? Regarding the latter point, two possibilities are convergence in probability, and almost sure convergence. Convergence in probability says that for a given $\epsilon > 0$, and sequence of single $n \times n$ GUE matrices ($n = 1, 2, \dots$), $\Pr(|\mu_n - \mu| > \epsilon) \rightarrow 0$, where μ_n is the empirical value of $\#[a, b]$ for each matrix, and μ is the limiting ensemble average (i.e. the value implied by the Wigner semi-circle law). Almost sure convergence says that the measure of the sequence of matrices for which

$\mu_n \rightarrow \mu$ is equal to 1. A well known consequence of the Borel-Cantelli lemma in probability theory (see e.g. [26]) is that almost sure convergence is equivalent to the statement that for a given $\epsilon > 0$, $\sum_{n=1}^{\infty} \Pr(|\mu_n - \mu| > \epsilon) < \infty$. Note that a necessary condition for this is that $\Pr(|\mu_n - \mu| > \epsilon) \rightarrow 0$, and thus almost sure convergence implies convergence in probability. To estimate $\Pr(|\mu_n - \mu| > \epsilon)$, the *Chebyshev inequality* [26]

$$\Pr(|\mu_n - \mu| > \epsilon) \leq \frac{\langle (\mu_n - \mu)^2 \rangle_{\text{GUE}}}{\epsilon^2}$$

can be employed. Hence for convergence in probability, it is sufficient that $\langle (\mu_n - \mu)^2 \rangle_{\text{GUE}} \rightarrow 0$ as $n \rightarrow \infty$, while for almost sure convergence, it is sufficient that $\sum_{n=1}^{\infty} \langle (\mu_n - \mu)^2 \rangle_{\text{GUE}} < \infty$.

In Section 1.7.1 and Exercises 1.7 q.1 the Wigner semi-circle law has been studied through its moments. We have shown that $\langle N^{-k-1} \text{Tr} X^{2k} \rangle_{\text{GUE}^*} \rightarrow m_{2k}$ where m_{2k} is the corresponding moment of the Wigner semi-circle law. To study convergence in probability and almost sure we thus must study

$$\text{Var}(N^{-k-1} \text{Tr} X^{2k}) := \langle (N^{-k-1} \text{Tr} X^{2k})^2 \rangle_{\text{GUE}^*} - \left(\langle N^{-k-1} \text{Tr} X^{2k} \rangle_{\text{GUE}^*} \right)^2.$$

Now, analogous to (1.111) we have

$$\langle (\text{Tr} X^{2k})^2 \rangle_{\text{GUE}^*} = \left\langle \sum_{\substack{i_1, \dots, i_{2k}=1 \\ j_1, \dots, j_{2k}=1}} z_{i_1 i_2} \cdots z_{i_{2k} i_1} z_{j_1 j_2} \cdots z_{j_{2k} j_1} \right\rangle.$$

Regarding i_1, \dots, i_{2k} as fixed, and taking into consideration (1.112), one sees [99]

$$\begin{aligned} \langle (\text{Tr} X^{2k})^2 \rangle_{\text{GUE}^*} &= \left\langle \sum_{i_1, \dots, i_{2k}=1} z_{i_1 i_2} \cdots z_{i_{2k} i_1} \right\rangle_{\text{GUE}^*} \left\langle \sum_{j_1, \dots, j_{2k}=1} z_{j_1 j_2} \cdots z_{j_{2k} j_1} \right\rangle_{\text{GUE}^*} \left(1 + \text{O}\left(\frac{1}{N^2}\right) \right) \\ &= \langle \text{Tr} X^{2k} \rangle_{\text{GUE}^*}^2 \left(1 + \text{O}\left(\frac{1}{N^2}\right) \right). \end{aligned}$$

It follows from this that $\sum_{N=1}^{\infty} \text{Var}(N^{-k-1} \text{Tr} X^{2k}) < \infty$, so we can conclude that almost sure convergence holds and so the Wigner semi-circle law is the limiting density of all sequences of GUE matrices, up to a set of measure zero.

We remark that almost sure convergence to the Wigner semi-circle law can also be established for sequences of matrices from the Gaussian β -ensemble, in the case of general $\beta > 0$ [49].

