2 Dimensional analysis

Before moving on to more 'sophisticated things', let us think a little about dimensional analysis and scaling. On the one hand these are trivial, and on the other they give a simple method for getting answers to problems that might otherwise be intractable. The idea behind dimensional analysis is very simple: *Any physical law must be expressible in any system of units that you use.* There are two consequences of this:

- One can often guess the answer just by thinking about what the dimensions of the answer should be, and then expressing the answer in terms of quantities that are known to have those dimensions[1].

- The scientifically interesting results are *always* expressible in terms of quantities that are *dimensionless*, not depending on the system of units that you are using.

One example of a dimensionless number relevant for fluid dynamics that we have already encountered in the introductory class is the *Reynolds number*, which quantifies the relative strength of viscous and inertial forces. Another example of dimensional analysis that we will study in detail is the solution to the diffusion equation for the spreading of a point source. The only relevant physical parameter is the diffusion constant D, which has dimensions of L^2T^{-1}. We denote this by writing $[D] = L^2T^{-1}$. Therefore the characteristic scale over which the solution varies after time t must be $\sqrt{Dt}$. This might seem like a rather simple result, but it expresses the essence of solutions to the diffusion equation. Of course, we will be able to solve the diffusion equation exactly, so this argument wasn't really necessary. In practice, however, we will rarely find useful exact solutions to the Navier-Stokes equations, and so dimensional analysis will often give us insight before diving into the mathematics or numerical simulations. Before formalising our approach, let us consider a few examples where simple dimensional arguments intuitively lead to interesting results.

2.1 The pendulum

This is a trivial problem that you know quite well. Consider a pendulum with length L and mass m, hanging in a gravitational field of strength g. What is the period of the pendulum? We need a way to construct a quantity with units of time involving these numbers. The only possible way to do this is with the combination $\sqrt{L/g}$. Therefore, we know immediately that

$$\tau = c\sqrt{L/g}. \tag{23}$$

This result might seem trivial to you, as you will probably remember (e.g., from a previous course) that $c = 2\pi$, if one solves the full dynamical problem for for small amplitude oscillations. However, the above formula works even for large amplitude oscillations.

2.2 Pythagorean theorem

Now we try to prove the Pythagorean theorem by dimensional analysis. Suppose you are given a right triangle, with hypotenuse length L and smallest acute angle ϕ. The area of

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[1]Be careful to distinguish between *dimensions* and *units*. For example mass (M), length (L) and time (T) are dimensions, and they can have different units of measurement (e. g. length may be in feet or meters)

the triangle is clearly

$$A = A(L, \phi). \tag{24}$$

Since ϕ is dimensionless, it must be that

$$A = L^2 f(\phi), \tag{25}$$

where f is some function we don't know.

Now the triangle can be divided into two little right triangles by dropping a line from the right angle which is perpendicular to the hypotenuse. The two right triangles have hypotenuses that happen to be the other two sides of our original right triangle, let's call them a and b. So we know that the areas of the two smaller triangles are $a^2 f(\phi)$ and $b^2 f(\phi)$ (where elementary geometry shows that the acute angle ϕ is the same for the two little triangles as the big triangle). Moreover, since these are all right triangles, the function f is the same for each. Therefore, since the area of the big triangle is just the sum of the areas of the little ones, we have

$$L^2 f = a^2 f + b^2 f,$$

or

$$L^2 = a^2 + b^2. \tag{26}$$

2.3 The gravitational oscillation of a star

It is known that the sun, and many other stars undergo some mode of oscillation. The question we might ask is how does the frequency of oscillation ω depend on the properties of that star? The first step is to identify the physically relevant variables. These are the density ρ, the radius R and the gravitational constant G (as the oscillations are due to gravitational forces). So we have

$$\omega = \omega(\rho, R, G). \tag{27}$$

The dimensions of the variables are $[\omega] = T^{-1}$, $[\rho] = ML^{-3}$, $[R] = L$ and $[G] = M^{-1}L^3 T^{-2}$. The only way we can combine these to give as a quantity with the dimensions of time, is through the relation

$$\omega = c\sqrt{G\rho}. \tag{28}$$

Thus, we see that the frequency of oscillation is proportional to the square root of the density and independent of the radius. The determination of c requires a real stellar observation, but we have already determined a lot of interesting details from dimensional analysis alone. For the sun, $\rho = 1400\text{kg/m}^3$, giving $\omega \sim 3 \times 10^{-4}\text{s}^{-1}$. The period of oscillation is approximately 1 hour, which is reasonable. However, for a neutron star ($\rho = 7 \times 10^{11}\text{kgm}^{-3}$) we predict $\omega \sim 7000\text{s}^{-1}$, corresponding to a period in the milli-second range.

2.4 The oscillation of a droplet

What happens if instead of considering a large body of fluid, such as a star, we consider a smaller body of fluid, such as a raindrop. Well, in this case we argue that surface tension γ

provides the relevant restoring force and we can neglect gravity. γ has dimensions of energy/area, so that $[\gamma] = MT^{-2}$. The only quantity we can now make with the dimensions of T^{-1} using our physical variables is

$$\omega = c\sqrt{\frac{\gamma}{\rho R^3}}, \tag{29}$$

which is not independent of the radius. For water $\gamma = 0.07\text{Nm}^{-1}$ giving us a characteristic frequency of 3Hz for a raindrop.

One final question we might ask ourselves before moving on is how big does the droplet have to be for gravity to have an effect? We reason that the crossover will occur when the two models give the same frequency of oscillation. Thus, when

$$\sqrt{\rho G} = \sqrt{\frac{\gamma}{\rho R^3}} \tag{30}$$

we find that

$$R_c \sim \left(\frac{\gamma}{\rho^2 G}\right)^{\frac{1}{3}} \tag{31}$$

This gives a crossover radius of about 10m for water.

2.5 Water waves

This is a subject we will deal with in greater detail towards the end of the course, but for now we look to obtain a basic understanding of the motion of waves on the surface of water. For example, how does the frequency of the wave depend on the wavelength λ? This is called the *dispersion relation*.

If the wavelength is long, we expect gravity to provide the restoring force, and the relevant physical variables in determining the frequency would appear to be the mass density ρ, the gravitational acceleration g and the wave number $k = 2\pi/\lambda$. The dimensions of these quantities are $[\rho] = ML^{-3}$, $[g] = LT^{-2}$ and $[k] = L^{-1}$. We can construct a quantity with the dimensions of T^{-1} through the relation

$$\omega = c\sqrt{gk}. \tag{32}$$

We see that the frequency of water waves is proportional to the square root of the wavenumber, in contrast to light waves for which the frequency is proportional to the wavenumber.

As with a droplet, we might worry about the effects of surface tension when the wavelength gets small. In this case we replace g with γ in our list of physically relevant variables. Given that $[\gamma] = MT^{-2}$, the dispersion relation must be of the form

$$\omega = c\sqrt{\gamma k^3/\rho}, \tag{33}$$

which is very different to that for gravity waves. If we look for a crossover, we find that the frequencies of *gravity waves* and *capillary waves* are equal when

$$k \sim \sqrt{\rho g/\gamma}. \tag{34}$$

This gives a wavelength of 1cm for water waves.

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