

# Lecture 11: Pricing and hedging derivative securities

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Last time, we discussed financial time series, modeled by either additive or multiplicative discrete stochastic processes, leading to either normal or lognormal limiting distributions, respectively. Today we will talk about derivative securities which “derive” value from different underlying asset. Unlike financial time series, which are difficult to model with any certainty, derivatives are much better suited for mathematical analysis, because the appropriate *relationship* between a derivative and its underlying asset can be predicted with greater confidence. Indeed, derivatives pricing and hedging is the area of finance where mathematics, rooted in random walks and stochastic processes, has had (arguably) the greatest impact on both theory and practice.

## 1 Static hedge and fair price

A **derivative security** is a financial instrument whose value is “derived” from the value of one or more **underlying assets**, which could be commodities (e.g. pork bellies), stocks, foreign exchange rates or more exotic variables (e.g. the average snowfall in Aspen, CO).

### 1.1 Problem statement

The static case considers a single time interval of trading. The underlying asset is  $x_t$  (stochastic, discrete like random walk). We are given:

1. the transition probability  $p(x, T|x_0, 0)$  where  $t = T$  is the maturity, ( $p$  would be lognormal distribution in multiplicative case), where it is assumed that the underlying asset follows a Markov process
2. the payoff function  $y(x_T)$ .

In reality, there are always transaction costs but here we are not worrying about these details.

The initial derivative value is  $w_0 = w(x_0, t = 0)$  and its value at maturity  $T$  is  $w_T = w(x_T, T) = y(x_T)$ . Thus, we have the value of the asset and the transition probability. The problem is to determine the price of the derivative  $w_t$ .

## 1.2 Fair price

In 1900, Bachelier introduced the **fair price**,  $\langle y(x_T) \rangle e^{-rT}$ , where  $x_t$  is the underlying asset, and we have included interest rate effects by discounting the time value of money ( $e^{-rT}$  factor). The fair price is the expected value of the payoff function at maturity, discounted by interest effects. However, there is a flaw in the argument and actually this is not the fair price.

Actually, the two (the underlying asset  $x_t$  and option  $w_t$ ) are correlated and a clever trader would take advantage of this fact. Black-Scholes (1973) and Merton (1973) might have been the first to consider this strategy. The idea is that one can “hedge” risk by trading both the underlying asset  $x_t$  and the option  $w_t$ . We could trade in such a way as to obtain a guaranteed payoff. Therefore, to determine the fair price we have to take this into account.

## 1.3 Hedged trading position

Define  $u(x, t)$ , the **trading position**, as

$$u(x, t) = w(x_t, t) - \phi x_t \quad (1)$$

where the right hand side means hold one unit of derivative,  $w(x_t, t)$ , and short  $\phi$  units of the underlying asset. The value of the initial investment  $u_0$  is

$$u_0 = u(x_0, 0) = w_0 - \phi x_0 \quad (2)$$

Assuming a static hedge, (fixed  $\phi$ ), the value of the net investment at maturity is

$$u_T = u(x_T, t) = y(x_T) - \phi x_T \quad (3)$$

Coming back to Bachelier, the only thing he left out was that the optimal strategy does not involve just the option itself, but consists of a portfolio in reality. If the option  $w$  only depends on  $x_T$  ( $x_T$  is independent of other market quantities), then we still apply Bachelier’s “fair-game” argument. The value of initial investment

$$\begin{aligned} u_0 &= \langle u_T \rangle e^{-rT} \\ &= e^{-rT} \int_0^\infty (y(x) - \phi x) p(x, T | x_0, 0) dx \\ &= e^{-rT} (\langle y(x_T) \rangle - \phi \langle x_T \rangle) \end{aligned}$$

where  $x$  is positive and so the limits of integration are from zero to infinity. This is just Bachelier and there is no expected return. It agrees with unhedged theory if  $\langle x_T \rangle = x_0 e^{-rT}$ . We now have our static hedge and  $\phi$  is sometimes called the **hedge ratio**. The  $w_0$  price is unknown and  $\phi$  is also undetermined. We have developed the theory first, now the solution procedure will be presented.

Note: We mention another perspective where  $w(x_t, t) = u(x_t, t) + \phi x_t$ . We might follow this strategy, called the **replicating portfolio**. This could be dangerous, in that hopefully the market will become diversified in its strategies. This is blamed for the Black Monday stock market crash in 1987 due to instability of this Black-Scholes (1973) trading strategy.

## 2 Optimal hedging to minimize risk

In the case of the forward contract, discussed last time, where the payoff  $y(x)$  is linear, then we can pick  $\phi$  to eliminate risk, or uncertainty, in  $u_T$ . More generally, we want to pick the hedge ratio  $\phi$  to make  $u_T$  as close to  $\langle u_T \rangle$  as possible.

### 2.1 Definition of risk

The risk can be defined in many ways (e.g. variance of  $u$ , 95th percentile, 4th moment of  $u$ ) and each definition would have a different strategy. For example, one could define risk by

$$\langle |u_T - \langle u_T \rangle|^m \rangle \quad (4)$$

If  $m = 2$ , it corresponds to variance, for  $m = 1$  absolute moment, etc. The larger  $m$  is, the more the sensitivity is to large losses. The second moment is the easiest to work with mathematically.

We want  $u_T = y(x_T) - \phi x_T$  to be close to  $\langle u_T \rangle = e^{rT} u_0 = \tilde{u}_0$ , where we have eliminated interest rate effects. Or equivalently, we want  $\tilde{u}_0 + \phi x_T$ , which is a linear function of  $x_T$ , to be close to  $y(x_T)$ , a nonlinear payoff function which depends on  $x_T$ . **The solution procedure boils down to linear regression.**

There is a nice graphical interpretation of the solution. It is weighted linear regression weighted by the transition probability. The slope  $\phi$  of the line

$$\phi = \frac{\Delta y}{\Delta x} \simeq \frac{\Delta w_T}{\Delta x_T} \quad (5)$$

Please see Prof. Bazant's IAP notes for graphs of the solution.

## 3 Bouchard-Sornette quadratic risk minimization

Suppose that the risk is defined in terms of the variance of the hedge position  $u_T$ .

$$R^2 = \text{var}(u_T) = \langle (u_T - \langle u_T \rangle)^2 \rangle \quad (6)$$

We want to find the optimal hedge solution, or strategy,  $\phi$ , which minimizes the risk, where here the definition of risk is the  $m = 2$  case from before. Therefore, we have

$$\frac{dR}{d\phi} = 0 \quad (7)$$

for  $\phi = \phi^*$ , where  $\phi^*$  is the optimal hedge ratio. The solution is obtained by just least squares fitting,

$$\begin{aligned} \phi^* &= \frac{\langle xy \rangle - \langle x \rangle \langle y \rangle}{\langle x^2 \rangle - \langle x \rangle^2} = \frac{\text{cov}(y, x)}{\text{var}(x)} \\ &= \text{corr}(y, x) \end{aligned}$$

The correlation coefficient,  $\text{corr}(y, x)$ , corresponds in our case to a correlation between  $x_T$  and  $y(x_T)$ . (Note: correlation has a range of  $-1$  to  $1$  with perfect anticorrelation corresponding to  $-1$ .) The fitted intercept in the least squares problem is

$$\begin{aligned} u_0^* &= \langle u_T \rangle - \phi \langle x \rangle \\ &= \frac{\langle x^2 \rangle \langle y \rangle - \langle x \rangle \langle xy \rangle}{\text{var}(x)} \end{aligned}$$

The two parameters we have determined are  $\phi^*$  and  $u_0$ .

Procedure: Pick  $\phi^*$  and  $u_0^* = u_0 e^{rT}$  to fit by least squares (weighted by transition probability of  $x_T$ , i.e. weighted least squares). This tells us:

1.  $\phi^*$  optimal hedge ratio
2. The optimal “fair price”

$$\begin{aligned} w_0^* &= e^{rT} \tilde{w}_0^* \\ &= e^{rT} (u_0^* + \phi^* \tilde{x}_0) \\ &= \tilde{u}_0^* e^{rT} + \phi^* x_0 \end{aligned}$$

where  $\tilde{w}_0^*$  is the best options price, and this optimal “fair price” protects us against arbitrage opportunity.

3. Equivalently tells us  $u_0$ , the replicating optimal portfolio. That is,  $u_0^* + \phi^* x_0$  for  $w_0$ , where  $u_0^*$ , the optimal amount of cash.

## 4 Binomial model

In the **binomial model**, the underlying asset can assume only two possible values, with probabilities  $p$  and  $q = 1 - p$ , where  $p$  is like the probability of “stepping to the right” in the binomial random walk.

$$p(x_T, T | x_0) = \begin{cases} p & \text{if } x_T = x_+ \\ q = 1 - p & \text{if } x_T = x_- \end{cases}$$

There is only one reasonable fit (linear) through two points. Therefore, it doesn’t matter what the probabilities are. This is for ANY risk definition  $R$  and ANY transition probability  $(p, q)$ . There is no uncertainty:  $p = .99$  or  $p = .01$  would have the same solution. This is a very pathological case, but theoretically very interesting. This implies we don’t have to study market data if we live in a binomial world. There is a unique price and unique trading strategy.

Actually, it is like linear interpolation in that the solution has to go through the two points. If there were three points, then the solution would depend on the probabilities. The linear interpolation equation is

$$y^*(x) = \frac{y_+(x - x_-) + y_-(x - x_+)}{x_+ - x_-} \quad (8)$$

where  $(x_+, y_+)$  and  $(x_-, y_-)$  are the two points.

In the next lecture, we will consider dynamic hedging as opposed to static hedging.

## References

- [1] Bouchaud J.-P. and M Potters, *Theory of Financial Risks*, Cambridge, 2000.
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- [4] Merton R.C., Theory of rational option pricing, *Bell Journal of Economics*, 4, 141-183, 1973.