18.366 Random Walks and Diffusion
Solutions to Problem Set 3

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1 Modified Kramers-Moyall Expansion

1.1 Preliminaries (also covered in lecture)

1.1.1 General form of expansion

Independence of steps of the random walker (i.e. absence of "memory" of magnitude and direction of previous steps at a current step) is expressed by the Chapman-Kolmogorov (Bachelier) equation:

$$P_{N+1}(x) = \int_{-\infty}^{+\infty} P_N(x-y)p(x, t+\tau|x-y, t)dy \tag{1}$$

which is a recursion for $P_N(x-y)$, the PDF that the walker is at point $x-y$ at step N (time $t = N\tau$), in terms of $p(x, t+\tau|x-y, t)$, the given PDF that the walker makes step $x-y \to x$ at step $N \to N+1$ (time $t \to t+\tau$).

We can formally (i.e. without regard to actual behavior of the functions) expand product $P_N(x-y)p(x, t+\tau|x-y, t)$ in powers of y around point x in Taylor series:

$$\begin{aligned} P_N(x-y)p(x, t+\tau|x-y, t) &= P_N(x)p(x+y, t+\tau|x, t) - y\frac{\partial}{\partial x}\left(P_N(x)p(x+y, t+\tau|x, t)\right) + \\ &\quad \frac{y^2}{2!}\frac{\partial^2}{\partial x^2}\left(P_N(x)p(x+y, t+\tau|x, t)\right) - ... \end{aligned} \tag{2}$$

The partial derivatives $\frac{\partial}{\partial x}$ are with respect to x with y fixed. This expansion is not very self-evident; for its justification it might help to consider $p(x, t+\tau|x-y, t)$ as a function of $x-y$ and y: $p(x, t+\tau|x-y, t) \equiv f_{t+\tau,t}(x-y, y)$; or to compare the expansion to a "regular" Taylor series of a function $f(x-y)$ around point x in powers of y:

$$f(x-y) = f(x) - yf'(x) + \frac{y^2}{2}f''(x) - ..., \tag{3}$$

where effectively x in the LHS gets replaced by $x+y$ in the RHS, and therefore in (2) $p(x, t+\tau|x-y, t)$ in the LHS has to be replaced by $p(x+y, t+\tau|x, t)$ in the RHS.

*The solution to problem 1 adapted from Marat Rvachev (2003). Problem 3 is adapted from the final project of Ken Gosier (2001), which became a master's thesis in finance at the Courant Institute, NYU in 2002.

Substituting (2) in (1), converting integral of sum into sum of integrals and changing order of differentiation:

$$\begin{aligned}
P_{N+1}(x) &= \int_{-\infty}^{+\infty} P_N(x)p(x+y,t+\tau|x,t)dy - \int_{-\infty}^{+\infty} y\frac{\partial}{\partial x}\left[P_N(x)p(x+y,t+\tau|x,t)\right]dy + ... \\
&= P_N(x)M_0(x,t,\tau) - \frac{\partial}{\partial x}\left[P_N(x)\int_{-\infty}^{+\infty} yp(x+y,t+\tau|x,t)dy\right] + ... \\
&= P_N(x) + \sum_{n=1}^{\infty}\left(-\frac{\partial}{\partial x}\right)^n \left[P_N(x)M_n(x,t,\tau)\frac{1}{n!}\right] \\
&= P_N(x) + \tau\sum_{n=1}^{\infty}\left(-\frac{\partial}{\partial x}\right)^n \left[P_N(x)D_n(x,t,\tau)\right]. \qquad (4)
\end{aligned}$$

In the last two equalities we used the notation $M_n(x,t,\tau) = \int_{-\infty}^{+\infty} p(x+y,t+\tau|x,t)y^n dy$ and $D_n(x,t,\tau) = \frac{1}{n!\tau}M_n(x,t,\tau)$. Transferring $P_N(x)$ in (4) to the LHS and dividing both sides of the equation by τ:

$$\frac{P_{N+1}(x) - P_N(x)}{\tau} = \sum_{n=1}^{\infty}\left(-\frac{\partial}{\partial x}\right)^n \left[P_N(x)D_n(x,t,\tau)\right] \qquad (5)$$

Since $\rho(x,N\tau) = P_N(x)$ for all N and x, we can formally substitute $P_N(x)$ for $\rho(x,t)$, $P_{N+1}(x)$ for $\rho(x,t+\tau)$ $(t = N\tau)$:

$$\frac{\rho(x,t+\tau) - \rho(x,t)}{\tau} = \sum_{n=1}^{\infty}\left(-\frac{\partial}{\partial x}\right)^n \left[\rho(x,t)D_n(x,t,\tau)\right] \qquad (6)$$

In the continuum limit ($\tau \to 0$), $\rho(x,t+\tau)$ can be expanded in powers of τ as

$$\rho(x,t+\tau) = \rho(x,t) + \tau\frac{\partial\rho(x,t)}{\partial t} + \frac{\tau^2}{2!}\frac{\partial^2\rho(x,t)}{\partial t^2} + ... \qquad (7)$$

Substituting (7) in (6) and rearranging terms, we get the final result:

$$\frac{\partial\rho}{\partial t} + \sum_{n=2}^{\infty}\frac{\tau^{n-1}}{n!}\frac{\partial^n\rho}{\partial t^n} = \sum_{n=1}^{\infty}\left(-\frac{\partial}{\partial x}\right)^n [D_n(x,t,\tau)\rho(x,t)]. \qquad (8)$$

1.1.2 Fokker-Planck equation

Given that in the continuum limit ($N \to \infty$, $\tau \to 0$) the moments scale as $M_1 \sim D_1\tau$, $M_2 \sim 2D_2\tau$ and $M_n \sim M_2^{n/2} = O(\tau^{n/2})$, we can estimate orders of $D_n(x,t,\tau)$ as $\tau \to 0$:

$$D_1(x,t,\tau) = \frac{M_1(x,t,\tau)}{\tau} \sim D_1 \sim O(1), \qquad (9)$$

$$D_2(x,t,\tau) = \frac{M_2(x,t,\tau)}{2!\tau} \sim D_2 \sim O(1), \qquad (10)$$

$$D_n(x,t,\tau) = \frac{M_n(x,t,\tau)}{n!\tau} \sim \frac{O(\tau^{n/2})}{n!\tau} \sim O(\tau^{n/2-1}), \qquad n > 2 \qquad (11)$$

Here we used definition of D_n: $D_n(x,t,\tau) = \frac{M_n(x,t,\tau)}{n!\tau}$. By inspecting equation (8) and discarding terms of orders of $\tau^{1/2}$ or higher[1], equation (8) can be rewritten as

$$\frac{\partial\rho}{\partial t} + O(\tau) = -\frac{\partial}{\partial x}(D_1\rho) + \frac{\partial^2}{\partial x^2}(D_2\rho) + O(\tau^{1/2}), \qquad (12)$$

[1]Here and below we assume that orders of derivatives of $(D_n\rho)$ and ρ are not lower (in τ) than the orders of $(D_n\rho)$ and ρ respectively, i.e. that $(D_n\rho)$ and ρ are smooth enough functions.

or

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(D_1\rho) = \frac{\partial^2}{\partial x^2}(D_2\rho) + O(\tau^{1/2}), \qquad \tau \to 0, \tag{13}$$

which is the required Fokker-Planck equation.

1.2 Modified Kramers-Moyall Expansion: Higher-order terms

By inspecting (8) and taking into account orders of D_n given by (9)-(11), it can be seen that the only term at $O(\tau^{1/2})$ is $(-\frac{\partial^3}{\partial x^3}(D_3\rho))$. Including this term in (13) we obtain:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(D_1\rho) = \frac{\partial^2}{\partial x^2}(D_2\rho) - \frac{\partial^3}{\partial x^3}(D_3\rho) + O(\tau), \qquad \tau \to 0. \tag{14}$$

Similarly, keeping in (8) terms of order of $O(\tau)$ or lower:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(D_1\rho) + \frac{\tau}{2}\frac{\partial^2 \rho}{\partial t^2} + O(\tau^2) = \frac{\partial^2}{\partial x^2}(D_2\rho) - \frac{\partial^3}{\partial x^3}(D_3\rho) + \frac{\partial^4}{\partial x^4}(D_4\rho) + O(\tau^{3/2}). \tag{15}$$

We will now eliminate the second time derivative of ρ in the third term in the LHS. Rearranging terms:

$$\frac{\partial \rho}{\partial t} = -\frac{\tau}{2}\frac{\partial^2 \rho}{\partial t^2} - \frac{\partial}{\partial x}(D_1\rho) + \frac{\partial^2}{\partial x^2}(D_2\rho) - \frac{\partial^3}{\partial x^3}(D_3\rho) + \frac{\partial^4}{\partial x^4}(D_4\rho) + O(\tau^{3/2}). \tag{16}$$

Applying $\frac{\tau}{2}\frac{\partial}{\partial t}$ to both sides and changing order of differentiation:

$$\frac{\tau}{2}\frac{\partial^2 \rho}{\partial t^2} = \frac{\tau}{2}\left[-\frac{\tau}{2}\frac{\partial^3 \rho}{\partial t^3} - \frac{\partial}{\partial x}\frac{\partial}{\partial t}(D_1\rho) + \frac{\partial^2}{\partial x^2}\frac{\partial}{\partial t}(D_2\rho) - \frac{\partial^3}{\partial x^3}\frac{\partial}{\partial t}(D_3\rho) + \frac{\partial^4}{\partial x^4}\frac{\partial}{\partial t}(D_4\rho)\right] + O(\tau^{3/2}). \tag{17}$$

Including terms of orders higher than or equal to $\tau^{3/2}$ in $O(\tau^{3/2})$ and taking some of the derivatives:

$$\begin{aligned}\frac{\tau}{2}\frac{\partial^2 \rho}{\partial t^2} &= \frac{\tau}{2}\left[-\frac{\partial}{\partial x}\frac{\partial}{\partial t}(D_1\rho) + \frac{\partial^2}{\partial x^2}\frac{\partial}{\partial t}(D_2\rho)\right] + O(\tau^{3/2}) \\ &= \frac{\tau}{2}\left[-\frac{\partial}{\partial x}\left(\rho\frac{\partial D_1}{\partial t} + D_1\frac{\partial \rho}{\partial t}\right) + \frac{\partial^2}{\partial x^2}\left(\rho\frac{\partial D_2}{\partial t} + D_2\frac{\partial \rho}{\partial t}\right)\right] + O(\tau^{3/2}).\end{aligned} \tag{18}$$

Substituting in (18) expression for $\frac{\partial \rho}{\partial t}$ from (16) and keeping only orders up to (and not including) $O(\tau^{1/2})$ in the square brackets:

$$\begin{aligned}\frac{\tau}{2}\frac{\partial^2 \rho}{\partial t^2} = &-\frac{\tau}{2}\frac{\partial}{\partial x}\left(\rho\frac{\partial D_1}{\partial t} + D_1\left[-\frac{\partial}{\partial x}(D_1\rho) + \frac{\partial^2}{\partial x^2}(D_2\rho)\right]\right) \\ &+\frac{\tau}{2}\frac{\partial^2}{\partial x^2}\left(\rho\frac{\partial D_2}{\partial t} + D_2\left[-\frac{\partial}{\partial x}(D_1\rho) + \frac{\partial^2}{\partial x^2}(D_2\rho)\right]\right) + O(\tau^{3/2}).\end{aligned} \tag{19}$$

Now plugging this last expression back into (16) we get the answer:

$$\begin{aligned}\frac{\partial \rho}{\partial t} = &\frac{\tau}{2}\frac{\partial}{\partial x}\left(\rho\frac{\partial D_1}{\partial t} + D_1\left[-\frac{\partial}{\partial x}(D_1\rho) + \frac{\partial^2}{\partial x^2}(D_2\rho)\right]\right) \\ &-\frac{\tau}{2}\frac{\partial^2}{\partial x^2}\left(\rho\frac{\partial D_2}{\partial t} + D_2\left[-\frac{\partial}{\partial x}(D_1\rho) + \frac{\partial^2}{\partial x^2}(D_2\rho)\right]\right) \\ &-\frac{\partial}{\partial x}(D_1\rho) + \frac{\partial^2}{\partial x^2}(D_2\rho) - \frac{\partial^3}{\partial x^3}(D_3\rho) + \frac{\partial^4}{\partial x^4}(D_4\rho) + O(\tau^{3/2}).\end{aligned} \tag{20}$$

The RHS of this equation involves only spatial derivatives of ρ. The large brackets could of course be opened; this, however, would not lead to significant cancellation of terms and simplification of the equation.

2 Black-Scholes Formulae for Options Prices

The appplication of the Black-Scholes equation to simple options is discussed in many textbooks using a variety of methods, e.g. Hull, *Options, Futures, and Other Derivative Securities.*

3 Continuum Limit of Bouchaud-Sornette Options Theory

For a general and thorough study of options pricing by minimizing the Bouchaud-Sornette "quadratic risk" allowing for a variety of discrete random walks for the underlying asset (including additive and multiplicative stochastic processes) and interest-rate effects, see the final project report (and Master's thesis) of Ken Gosier, *Pricing and Hedging of Residual Risk* (2001), available at the class web site. The answer to this question is contained there, scattered throughout the paper, and the main part is reproduced below.

Suppose the underlying asset follows a discrete random walk with (additive) independent steps. Assume the displacements $y = \delta x$ in each time step δt have low order moments which depend on the current price,

$$\begin{aligned} \langle \delta x \rangle &= \mu x \delta t & (21) \\ \langle \delta x^2 \rangle &= \sigma^2 x^2 \delta t + \mu^2 x^2 \delta t^2 & (22) \\ \langle \delta x^3 \rangle &= \sigma^3 \lambda_3 x^3 \delta t^{3/2} + 3\mu\sigma^2 x^3 \delta t^2 + O(\delta t^3) & (23) \\ \langle \delta x^4 \rangle &= \sigma^4 (\lambda_4 + 3) x^4 \delta t^2 + O(\delta t^{5/2}). & (24) \end{aligned}$$

The random displacement is proprotional to the current value, so this is a discrete approximation of a multiplicative stochastic process (as is assumed in Black-Scholes theory). The time step δt could be viewed as the correlation time of the underlying asset, beyond which price increments are effectively independent.

Next we follow the Bouchaud-Sornette strategy of minimizing the "quadratic risk", or variance of the return of a position consition of the option and short ϕ of the underlying. Since minimizing the total variance is equivalent to minimizing the variance in each time step, we get the least-squares fit equations given in class, as a recursion for $w(x, t)$.

$$u(x,t) = w(x,t) - \phi(x,t)x = e^{-r\delta t} \left[\int w(x+\delta x, t+\delta t) p(\delta x, \delta t) d\delta x - \phi(x,t) \int (x+\delta x) p(\delta x, \delta t) d\delta x \right] \quad (25)$$

and

$$\phi^*(x,t) = \frac{1}{\sigma^2 x^2 \delta t} \int (\delta x - \mu x \delta t) w(x+\delta x, t+\delta t) p(\delta x, \delta t) d\delta x \quad (26)$$

The optimal hedge ratio ϕ^* in Bouchaud-Sornette theory is given by the slope of the "least-squares fit" of the hedged position to the option price, as explained in class, so $\phi^* = \mathrm{Cov}(x, w)/\mathrm{Var}(x) = \mathrm{Cov}(\delta x, w)/\mathrm{Var}(\delta x)$.

After specifying the low-order moments of the random walk for the underlying asset, Eqs. (21)–(24), the fair game argument, Eq. (25), and the Bouchaud-Sornette hedging strategy, Eq. (26), we now have all the ingredients necessary to derive the pricing PDE for the option w. We will show that this PDE agrees with the Black-Scholes equation to leading-order, but also contains correction terms. The size of the corrections will be proportional to the discrete time step δt, the parameters $\mu, \sigma, \lambda_3, \lambda_4$ from the moments, Eqs. (21)–(24), and the risk-free interest rate.

We begin by formally deriving moment expansions for terms on the right-hand side of Eq. (25). For the integral in w, we approximate the function by its Taylor series.

$$\begin{aligned} w(x+\delta x, t+\delta t) &= w(x,t) \qquad (27)\\ &+ \frac{\partial w}{\partial x}(x,t)\delta x + \frac{\partial w}{\partial t}(x,t)\delta t \\ &+ \frac{1}{2}\frac{\partial^2 w}{\partial x^2}(x,t)\delta x^2 + \frac{1}{2}\frac{\partial^2 w}{\partial t^2}(x,t)\delta t^2 + \frac{\partial^2 w}{\partial x \partial t}(x,t)\delta x \delta t + \cdots \qquad (28)\end{aligned}$$

In making these expansions, we assume that the function $w(x,t)$ is slowly varying at length and time scales much larger than those of individual steps, so that it is differentiable in the limit $\delta t \to 0$. This is a reasonable assumption as long as the maturity of the option is much larger than the time step.

Substituting Eq. (27) into the w integral in Eq. (25), we obtain an expression in which we have the derivatives for w at the initial coordinates (x,t), integrated against powers of the increments δx and δt. The derivatives are constants as far as the integral is concerned. Using the moments of δx, we obtain

$$\begin{aligned} & e^{-r\delta t}\int w(x+\delta x, t+\delta t) p\left(\frac{\delta x}{x}, \delta t\right) d\left(\frac{\delta x}{x}\right) \qquad (29)\\ & = w \\ & +(\delta t)\left[\mu x \frac{\partial w}{\partial x} + \frac{\partial w}{\partial t} + \frac{\sigma^2 x^2}{2}\frac{\partial^2 w}{\partial x^2} - rw\right] \\ & +(\delta t^{3/2})\left[\frac{\sigma^3 \lambda_3 x^3}{6}\frac{\partial^3 w}{\partial x^3}\right] \\ & +(\delta t^2)\left[\frac{\mu^2 x^2}{2}\frac{\partial^2 w}{\partial x^2} + \frac{1}{2}\frac{\partial^2 w}{\partial t^2} + \mu x \frac{\partial^2 w}{\partial x \partial t} + \frac{\mu \sigma^2 x^3}{2}\frac{\partial^3 w}{\partial x^3} + \frac{\sigma^2 x^2}{2}\frac{\partial^3 w}{\partial x^2 \partial t} + \frac{\sigma^4(\lambda_4+3)x^4}{24}\frac{\partial^4 w}{\partial x^4}\right] \\ & -(r\delta t^2)\left[\mu x \frac{\partial w}{\partial x} + \frac{\partial w}{\partial t} + \frac{\sigma^2 x^2}{2}\frac{\partial^2 w}{\partial x^2} - \frac{r}{2}w\right] \\ & +O(\delta t^{5/2}) \end{aligned}$$

where we have also approximated the exponential term $e^{-r\delta t}$ by its Taylor series. We leave the terms at order-δt^2 split out because of a simplification that will be made later. Next we perform a similar procedure for ϕ in Eq. (26), which yields,

$$\begin{aligned} \phi^* &= \frac{\partial w}{\partial x} \qquad (30)\\ &+ (\delta t^{1/2})\left[\frac{\sigma \lambda_3 x}{2}\frac{\partial^2 w}{\partial x^2}\right] \\ &+ (\delta t)\left[\mu x \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial x \partial t} + \frac{\sigma^2(\lambda_4+3)x^2}{6}\frac{\partial^3 w}{\partial x^3}\right] \\ &+ O(\delta t^{3/2}). \end{aligned}$$

Thus we see that ϕ^* agrees with the Black-Scholes hedge $\partial w/\partial x$ to leading-order, but also contains correction terms at higher order in δt.

With the approximations Eqs. (29) and (30) in hand, we re-write the fair-game equation, Eq. (25) and obtain (after considerable algebra), the expansion

$$0 = \left[rx\frac{\partial w}{\partial x} + \frac{\partial w}{\partial t} + \frac{\sigma^2 x^2}{2}\frac{\partial^2 w}{\partial x^2} - rw\right] \qquad (31)$$

$$
\begin{aligned}
&+ \quad (\delta t^{1/2}) \left[\frac{\sigma^3 \lambda_3 x^3}{6} \frac{\partial^3 w}{\partial x^3} + \frac{(r-\mu)\sigma\lambda_3 x^2}{2} \frac{\partial^2 w}{\partial x^2} \right] \\
&+ \quad (\delta t)[\left(r - \frac{\mu}{2} \right) \mu x^2 \frac{\partial^2 w}{\partial x^2} + \frac{1}{2} \frac{\partial^2 w}{\partial t^2} + rx \frac{\partial^2 w}{\partial x \partial t} + \left(\frac{(r-\mu)\sigma^2 \lambda_4 x^3}{6} + \frac{r\sigma^2 x^3}{2} \right) \frac{\partial^3 w}{\partial x^3} + \frac{\sigma^2 x^2}{2} \frac{\partial^3 w}{\partial x^2 \partial t} \\
&\qquad + \frac{\sigma^4 (\lambda_4 + 3) x^4}{24} \frac{\partial^4 w}{\partial x^4}] \\
&- \quad (r\delta t) \left[\frac{rx}{2} \frac{\partial w}{\partial x} + \frac{\partial w}{\partial t} + \frac{\sigma^2 x^2}{2} \frac{\partial^2 w}{\partial x^2} - \frac{r}{2} w \right] \\
&+ \quad O(\delta t^{3/2}).
\end{aligned}
$$

We may simplify the expression at order-δt by a series of recursive substitutions. These substitutions basically rely on the fact that a single time derivative may be replaced by a combination of spatial derivatives plus terms at higher-order in δt. This may be seen by writing out the $O(1)$ term of (31) as

$$
\frac{\partial w}{\partial t} = -rx \frac{\partial w}{\partial x} - \frac{\sigma^2 x^2}{2} \frac{\partial^2 w}{\partial x^2} + rw + O(\delta t^{1/2}) \tag{32}
$$

Showing the details of the first recursive substitution, we have

$$
(-r\delta t) \left[\frac{rx}{2} \frac{\partial w}{\partial x} + \frac{\partial w}{\partial t} + \frac{\sigma^2 x^2}{2} \frac{\partial^2 w}{\partial x^2} - \frac{r}{2} w \right] = (\delta t) \left[\frac{r^2 x}{2} \frac{\partial w}{\partial x} - \frac{r^2}{2} w \right] + O(\delta t^{3/2}). \tag{33}
$$

This substitution, and one more like it, lead to the final form of the pricing PDE.

$$
\begin{aligned}
0 \quad = \quad & \left[rx \frac{\partial w}{\partial x} + \frac{\partial w}{\partial t} + \frac{\sigma^2 x^2}{2} \frac{\partial^2 w}{\partial x^2} - rw \right] \qquad (34) \\
+ \quad & (\delta t^{1/2}) \left[\frac{\sigma^3 \lambda_3}{6} x^3 \frac{\partial^3 w}{\partial x^3} + \frac{(r-\mu)\sigma\lambda_3}{2} x^2 \frac{\partial^2 w}{\partial x^2} \right] \\
+ \quad & (\delta t) \left[\lambda_4 \left(\frac{\sigma^4}{24} x^4 \frac{\partial^4 w}{\partial x^4} + \frac{(r-\mu)\sigma^2}{6} x^3 \frac{\partial^3 w}{\partial x^3} \right) - x^3 \frac{\partial^3 w}{\partial x^3} \left(\frac{\sigma^4}{2} \right) - x^2 \frac{\partial^2 w}{\partial x^2} \left(\frac{(r-\mu)^2}{2} + r\sigma^2 + \frac{\sigma^4}{4} \right) \right] \\
+ \quad & O(\delta t^{3/2})
\end{aligned}
$$

This PDE agrees with Black-Scholes at leading-order, but also contains correction terms proportional to powers of the time step δt and the parameters of the underlying probability distribution, in particular the mean μ. Therefore, contrary to the interpretation in problem 2, the "optimal" options price generally does not correspond to a risk-neutral valuation.

The correction terms above were assigned as extra credit. For the solution to the "extra extra credit" – to solve the perturbed Black-Scholes PDE for a call option – see Gosier (2001), who starts by deriving the Green function.