

Midterm Exam

Due at 1pm in lecture on Thursday November 16, 2006.

Directions: (i) Work independently and not discuss the exam with anyone. (ii) You may quote formulae from 2005 18.366 lecture notes and problem sets, but otherwise all steps in your solutions should be derived in detail. (iii) You may consult the required and recommended books, but no other resources (books, web sites, etc.). Total exam time: 46.5 hours.

1. Discrete versus continuous steps in a random walk.

(a) Consider the probability distribution

$$p_n = C(a)a^{|n|}, \quad (0 < a < 1)$$

for displacements Δx in a random walk on the integers on the integers, and find its generating function

$$P(z) = \sum_{n=-\infty}^{\infty} p_n z^n.$$

Choose $C(a)$ to set $P(1) = 1$. Let $P_{N,n}$ be the probability distribution for the position after N IID steps. Use $P(z)$ to compute the diffusivity $D_2(a)$ which arises in the Fokker-Planck equation,

$$\frac{\partial \rho}{\partial t} = D_2 \frac{\partial^2 \rho}{\partial x^2}$$

for the continuum approximation $\rho(x = n, t = N) = P_{N,n}$ for $N \gg 1$ and $|n| = O(\sqrt{N})$.

(b) Now consider continuous displacements with the PDF,

$$p(x) = \frac{1}{2b} e^{-|x|/b}, \quad (b = -1/\log a)$$

and find its Fourier transform,

$$\hat{p}(k) = \int_{-\infty}^{\infty} e^{-ikx} p(x) dx.$$

Let $P_N(x)$ be the PDF for the position after N IID steps. Use $\hat{p}(k)$ to obtain *all* the coefficients $\bar{D}_{2m}(b)$ arising in the (modified) Kramers-Moyall expansion,

$$\frac{\partial \rho}{\partial t} = \sum_{m=1}^{\infty} \bar{D}_{2m} \frac{\partial^{2m} \rho}{\partial x^{2m}},$$

for the continuum approximation $\rho(x, t = N) = P_N(x)$ for $N \gg 1$ and $|n| = O(\sqrt{N})$.

(c) Make a log-linear plot of $D_2(a)$ and $\bar{D}_2(b(a))$ versus a , and explain in what limit they are the same.

2. **First passage of N random walks in two dimensions.** Consider continuous diffusion processes ($\rho_t = D\nabla^2\rho$) approximating random walks in the plane starting at $(x, y) = (0, a)$.

- (a) Let $G(x, t|x_0 = 0)$ be the Green function for the x component (marginal probability density) and $f(t|a)$ be the first passage time density for each walker. Compute

$$\varepsilon(x|a) = \int_0^\infty f(t|a)G(x, t|0)dt$$

and explain why this is the PDF of the location of first passage X (which was derived in lecture by conformal mapping). Note that $\langle X^2 \rangle = \infty$.

- (b) Now suppose N independent random walkers are released from $(0, a)$ at the same time, and compute¹ the PDF $\varepsilon_N(x|a)$ of the location X_N where, for the first time, one of these walkers hits the line. What is the critical number of walkers N_c such that $\langle X_N^2 \rangle < \infty$ for $N \geq N_c$?

3. **First passage to a circle.** Using conformal mapping² or the method of images, derive the PDF of the angle of first passage $\varepsilon(\theta)$ to the unit circle from the point $(a, 0)$ by continuous diffusion in the plane ($a > 0, a \neq 1$). What is the ratio of hitting probabilities for the nearest and farthest points, $\varepsilon(0)/\varepsilon(\pi)$?

Extra credit: Give a simple geometrical interpretation of $\varepsilon(\theta)$.

¹You may leave your answer as an integral, but to get full credit, you should “compute” $\varepsilon_N(x|a)$ numerically and plot versus x for say $N = 1, 2, 3$.

²Hint: use a Möbius transformation, $w = f(z) = (az + b)/(cz + d)$ to map the domain to the interior of the unit circle, with the source at the origin. This requires picking a, b, c, d such that, e.g. $w(a) = 0, w(1) = 1, \dots$