Last time:
- $A_{TM}$ is undecidable
- The diagonalization method
- $A_{TM}$ is T-unrecognizable
- The Reducibility Method, preview

Today: (Sipser §5.1, §5.3)
- The Reducibility Method for proving undecidability and T-unrecognizability.
- General reducibility
- Mapping reducibility
The Reducibility Method

If we know that some problem (say $A_{TM}$) is undecidable, we can use that to show other problems are undecidable.

**Defn:** $HALT_{TM} = \{(M, w) | M \text{ halts on input } w\}$

**Recall Theorem:** $HALT_{TM}$ is undecidable

Proof by contradiction, showing that $A_{TM}$ is reducible to $HALT_{TM}$:

Assume that $HALT_{TM}$ is decidable and show that $A_{TM}$ is decidable (false!).

Let TM $R$ decide $HALT_{TM}$.

Construct TM $S$ deciding $A_{TM}$.

$S = \{\text{On input } \langle M, w \rangle \}$

1. Use $R$ to test if $M$ on $w$ halts. If not, reject.
2. Simulate $M$ on $w$ until it halts (as guaranteed by $R$).
3. If $M$ has accepted then accept.
   If $M$ has rejected then reject.

TM $S$ decides $A_{TM}$, a contradiction. Therefore $HALT_{TM}$ is undecidable.
Reducibility – Concept

If we have two languages (or problems) $A$ and $B$, then $A$ is reducible to $B$ means that we can use $B$ to solve $A$.

**Example 1:** Measuring the area of a rectangle is reducible to measuring the lengths of its sides.

**Example 2:** We showed that $A_{\text{NFA}}$ is reducible to $A_{\text{DFA}}$.

**Example 3:** From Pset 2, $PUSHER$ is reducible to $E_{\text{CFG}}$. (Idea- Convert push states to accept states.)

If $A$ is reducible to $B$ then solving $B$ gives a solution to $A$.

- then $B$ is easy $\rightarrow$ $A$ is easy.
- then $A$ is hard $\rightarrow$ $B$ is hard.

this is the form we will use

**Check-in 9.1**

Is Biology reducible to Physics?

(a) Yes, all aspects of the physical world may be explained in terms of Physics, at least in principle.

(b) No, some things in the world, maybe life, the brain, or consciousness, are beyond the realm of Physics.

(c) I’m on the fence on this question!
**$E_{TM}$ is undecidable**

Let $E_{TM} = \{(M) | M \text{ is a TM and } L(M) = \emptyset \}$

**Theorem:** $E_{TM}$ is undecidable

Proof by contradiction. Show that $A_{TM}$ is reducible to $E_{TM}$.

Assume that $E_{TM}$ is decidable and show that $A_{TM}$ is decidable (false!).

Let TM $R$ decide $E_{TM}$.

Construct TM $S$ deciding $A_{TM}$.

$S = \text{"On input } \langle M, w \rangle$

1. Transform $M$ to new TM $M_w = \text{"On input } x$
   1. If $x \neq w$, reject.
   2. else run $M$ on $w$
   3. Accept if $M$ accepts.”

2. Use $R$ to test whether $L(M_w) = \emptyset$

3. If YES [so $M$ rejects $w$] then reject.
   If NO [so $M$ accepts $w$] then accept.

$M_w$ works like $M$ except that it always rejects strings $x$ where $x \neq w$.

So $L(M_w) = \begin{cases} \{w\} & \text{if } M \text{ accepts } w \\ \emptyset & \text{if } M \text{ rejects } w \end{cases}$
**Mapping Reducibility**

**Defn:** Function \( f \colon \Sigma^* \rightarrow \Sigma^* \) is **computable** if there is a TM \( F \) where \( F \) on input \( w \) halts with \( f(w) \) on its tape, for all strings \( w \).

**Defn:** \( A \) is mapping-reducible to \( B \) (\( A \leq_m B \)) if there is a computable function \( f \) where \( w \in A \) iff \( f(w) \in B \).

**Example:** \( A_{TM} \leq_m \overline{E_{TM}} \)

The computable reduction function \( f \) is \( f(\langle M, w \rangle) = \langle M_w \rangle \)

Because \( \langle M, w \rangle \in A_{TM} \) iff \( \langle M_w \rangle \in \overline{E_{TM}} \)

( \( M \) accepts \( w \) iff \( L(\langle M_w \rangle) \neq \emptyset \) )

Recall TM \( M_w = \) “On input \( x \)

1. If \( x \neq w \), reject.
2. else run \( M \) on \( w \)
3. Accept if \( M \) accepts.”
Theorem: If $A \leq_m B$ and $B$ is decidable then so is $A$

Proof: Say TM $R$ decides $B$.

Construct TM $S$ deciding $A$:

$S =$ “On input $w$

1. Compute $f(w)$
2. Run $R$ on $f(w)$ to test if $f(w) \in B$
3. If $R$ halts then output same result.”

Corollary: If $A \leq_m B$ and $A$ is undecidable then so is $B$

Theorem: If $A \leq_m B$ and $B$ is T-recognizable then so is $A$

Proof: Same as above.

Corollary: If $A \leq_m B$ and $A$ is T-unrecognizable then so is $B$
Mapping vs General Reducibility

Mapping Reducibility of $A$ to $B$: Translate $A$-questions to $B$-questions.
- A special type of reducibility
- Useful to prove $T$-unrecognizability

(General) Reducibility of $A$ to $B$: Use $B$ solver to solve $A$.
- May be conceptually simpler
- Useful to prove undecidability

Noteworthy difference:
- $A$ is reducible to $\overline{A}$
- $A$ may not be mapping reducible to $\overline{A}$.
  For example $\overline{A_{TM}} \not\leq_{m} A_{TM}$

Check-in 9.3
We showed that if $A \leq_{m} B$ and $B$ is $T$-recognizable then so is $A$.
Is the same true if we use general reducibility instead of mapping reducibility?
(a) Yes
(b) No
Reducibility – Templates

To prove $B$ is undecidable:
- Show undecidable $A$ is reducible to $B$. (often $A$ is $A_{TM}$)
- Template: Assume TM $R$ decides $B$.
  Construct TM $S$ deciding $A$. Contradiction.

To prove $B$ is T-unrecognizable:
- Show T-unrecognizable $A$ is mapping reducible to $B$. (often $A$ is $A_{TM}$)
- Template: give reduction function $f$. 
Recall \( E_{TM} = \{ \langle M \rangle | M \text{ is a TM and } L(M) = \emptyset \} \)

**Theorem:** \( E_{TM} \) is T-unrecognizable

Proof: Show \( A_{TM} \leq_{m} E_{TM} \)

Reduction function: \( f(\langle M, w \rangle) = \langle M_w \rangle \)

Recall TM \( M_w = “\text{On input } x \) 1. If \( x \neq w \), reject.
2. Else run \( M \) on \( w \)
3. Accept if \( M \) accepts.”

Explanation: \( \langle M, w \rangle \in A_{TM} \) iff \( \langle M_w \rangle \in E_{TM} \)
\( M \) rejects \( w \) iff \( L(\langle M_w \rangle) = \emptyset \)
$E_{\text{TM}}$ and $\overline{E_{\text{TM}}}$ are T-unrecognizable

$E_{\text{TM}} = \{ \langle M_1, M_2 \rangle \mid M_1$ and $M_2$ are TMs and $L(M_1) = L(M_2) \}$

**Theorem:** Both $E_{\text{TM}}$ and $\overline{E_{\text{TM}}}$ are T-unrecognizable

**Proof:**

1. $A_{\text{TM}} \leq_m E_{\text{TM}}$
2. $A_{\text{TM}} \leq_m \overline{E_{\text{TM}}}$

For any $w$ let $T_w = \text{"On input } x \text{ \quad } T_w \text{ acts on all inputs the way } M \text{ acts on } w.$
   1. Ignore $x$.
   2. Simulate $M$ on $w$.

(1) Here we give $f$ which maps $A_{\text{TM}}$ problems (of the form $\langle M, w \rangle$) to $E_{\text{TM}}$ problems (of the form $\langle T_1, T_2 \rangle$).
   
   $f(\langle M, w \rangle) = \langle T_w, T_{\text{reject}} \rangle \quad T_{\text{reject}} \text{ is a TM that always rejects.}$

(2) Similarly $f(\langle M, w \rangle) = \langle T_w, T_{\text{accept}} \rangle \quad T_{\text{accept}} \text{ always accepts.}$
Reducibility terminology

Why do we use the term “reduce”? When we reduce $A$ to $B$, we show how to solve $A$ by using $B$ and conclude that $A$ is no harder than $B$. (suggests the $\leq_m$ notation)

Possibility 1: We bring $A$’s difficulty down to $B$’s difficulty.
Possibility 2: We bring $B$’s difficulty up to $A$’s difficulty.
Quick review of today

1. Introduced The Reducibility Method to prove undecidability and T-unrecognizability.
2. Defined mapping reducibility as a type of reducibility.
3. $E_{TM}$ is undecidable.
4. $E_{TM}$ is T-unrecognizable.
5. $EQ_{TM}$ and $EQ_{TM}$ are T-unrecognizable.