MASSACHUSETTS INSTITUTE OF TECHNOLOGY

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Quantum Computation
Fall 2004

Problem Set 8
Solutions

Problem 1. Write out

$$U = e^{-i\pi(\sigma_X \otimes I)/4} e^{-i\pi(\sigma_Z \otimes \sigma_Z)/4} e^{-i\pi(\sigma_Y \otimes I)/4}$$

as a 4 by 4 matrix . How does it differ from a CNOT gate? How can you supplement the above circuit with single-qubit rotations to construct an exact CNOT (up to an overall phase)?

Solution:

$$\begin{aligned}
U &= e^{-i\pi(\sigma_X \otimes I)/4} e^{-i\pi(\sigma_Z \otimes \sigma_Z)/4} e^{-i\pi(\sigma_Y \otimes I)/4} \\
&= \frac{1}{2\sqrt{2}}(I - i\sigma_X \otimes I)(I - i\sigma_Z \otimes \sigma_Z)(I - i\sigma_Y \otimes I) \\
&= \frac{1}{2\sqrt{2}}(I - i\sigma_X \otimes I - i\sigma_Z \otimes \sigma_Z + i\sigma_Y \otimes \sigma_Z)(I - i\sigma_Y \otimes I) \\
&= \frac{1}{2\sqrt{2}}(I - i\sigma_X \otimes I - i\sigma_Z \otimes \sigma_Z + i\sigma_Y \otimes \sigma_Z \\
&\qquad -i\sigma_Y \otimes I - i\sigma_Z \otimes I + i\sigma_X \otimes \sigma_Z + I \otimes \sigma_Z) \\
&= \frac{-i}{2\sqrt{2}}[(iI + \sigma_X + \sigma_Y + \sigma_Z) \otimes I + (\sigma_Z - \sigma_Y - \sigma_X + iI) \otimes \sigma_Z]
\end{aligned}$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 1-i & 0 & 0 & 0 \\ 0 & 0 & 0 & -1-i \\ 0 & 0 & i+1 & 0 \\ 0 & -1+i & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} e^{-i\pi/4} & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{-i3\pi/4} \\ 0 & 0 & e^{i\pi/4} & 0 \\ 0 & e^{-i\pi/4} & 0 & 0 \end{bmatrix}$$

which looks like a CONT with the second qubit as the control bit except for the wrong phases.

To correct the phases, we can use $\exp(i\theta I\otimes\sigma_Z)$ and $\exp(i\phi\sigma_Z\otimes I)$. We have

$$\exp(i\theta I\otimes\sigma_Z)=\begin{bmatrix} e^{i\theta} & 0 & 0 & 0 \\ 0 & e^{-i\theta} & 0 & 0 \\ 0 & 0 & e^{i\theta} & 0 \\ 0 & 0 & 0 & e^{-i\theta} \end{bmatrix}$$

and

$$\exp(i\phi\sigma_Z\otimes I)=\begin{bmatrix} e^{i\phi} & 0 & 0 & 0 \\ 0 & e^{i\phi} & 0 & 0 \\ 0 & 0 & e^{-i\phi} & 0 \\ 0 & 0 & 0 & e^{-i\phi} \end{bmatrix}$$

Therefore,

$$e^{i\theta I\otimes\sigma_Z}\,e^{i\phi\sigma_Z\otimes I}U=\begin{bmatrix} e^{+i(\theta+\phi-\pi/4)} & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{-i(\theta-\phi+3\pi/4)} \\ 0 & 0 & e^{i(\theta-\phi+\pi/4)} & 0 \\ 0 & e^{i(-\pi/4-\theta-\phi)} & 0 & 0 \end{bmatrix}$$

which is a CNOT for $\theta=3\phi=3\pi/4$, up to an overall phase $\exp(i3\pi/4)$.

Problem 2. (Decoupling). Show that

$$U=e^{-i\pi(\sigma_X\otimes I)/2}e^{-iHt}e^{-i\pi(\sigma_X\otimes I)/2}e^{-iHt}$$

only acts on the second qubit, where

$$H=\frac{1}{2}(-\omega_1\sigma_Z\otimes I-\omega_2 I\otimes\sigma_Z+\gamma\sigma_Z\otimes\sigma_Z).$$

Hint 1. If $[H_1,H_2]=0$, then $\exp(-i(H_1+H_2)t)=\exp(-iH_1t)\exp(-iH_2t)$.
Hint 2. You may find it useful to first find the following operations $(\sigma_X\otimes I)\exp(-i\omega t(\sigma_Z\otimes I)/2)(\sigma_X\otimes I)$, $(\sigma_X\otimes I)\exp(-i\omega t(I\otimes\sigma_Z)/2)(\sigma_X\otimes I)$, and $(\sigma_X\otimes I)\exp(-i\omega t(\sigma_Z\otimes\sigma_Z)/2)(\sigma_X\otimes I)$.

Solution:

It can be seen that

$$(\sigma_X\otimes I)\exp(-i\omega t(\sigma_Z\otimes I)/2)(\sigma_X\otimes I)=\cos(\omega t/2)\sigma_X^2\otimes I$$

$$-i\sin(\omega t/2)\sigma_X\sigma_Z\sigma_X \otimes I$$
$$= \cos(\omega t/2) I \otimes I + i\sin(\omega t/2)\sigma_Z \otimes I$$
$$= \exp(i(\omega t/2)\sigma_Z \otimes I).$$

Similarly,

$$(\sigma_X \otimes I)\exp(-i\omega t(\sigma_Z \otimes \sigma_Z)/2)(\sigma_X \otimes I) = \exp(i\omega t(\sigma_Z \otimes \sigma_Z)/2).$$

Also, since $[\sigma_X \otimes I, I \otimes \sigma_Z] = 0$, we have

$$(\sigma_X \otimes I)\exp(-i\omega t(I \otimes \sigma_Z)/2)(\sigma_X \otimes I) = \exp(-i\omega t(I \otimes \sigma_Z)/2).$$

Now, since $\sigma_Z \otimes I$, $I \otimes \sigma_Z$, and $\sigma_Z \otimes \sigma_Z$ commute with each other, using the above three equations, we have,

$$\begin{aligned}
U &= e^{-i\pi(\sigma_X \otimes I)/2} e^{-iHt} e^{-i\pi(\sigma_X \otimes I)/2} e^{-iHt} \\
&= -(\sigma_X \otimes I) e^{i\omega_1 t(\sigma_Z \otimes I)/2} e^{i\omega_2 t(I \otimes \sigma_Z)/2} e^{-i\gamma t(\sigma_Z \otimes \sigma_Z)/2} (\sigma_X \otimes I) e^{-iHt} \\
&= -(\sigma_X \otimes I) e^{i\omega_1 t(\sigma_Z \otimes I)/2} (\sigma_X \otimes I)(\sigma_X \otimes I) e^{i\omega_2 t(I \otimes \sigma_Z)/2} (\sigma_X \otimes I) \\
&\qquad\qquad (\sigma_X \otimes I) e^{-i\gamma t(\sigma_Z \otimes \sigma_Z)/2} (\sigma_X \otimes I) e^{-iHt} \\
&= -e^{-i\omega_1 t(\sigma_Z \otimes I)/2} e^{i\omega_2 t(I \otimes \sigma_Z)/2} e^{i\gamma t(\sigma_Z \otimes \sigma_Z)/2} e^{i\omega_1 t(\sigma_Z \otimes I)/2} e^{i\omega_2 t(I \otimes \sigma_Z)/2} e^{-i\gamma t(\sigma_Z \otimes \sigma_Z)/2} \\
&= -e^{i\omega_2 t(I \otimes \sigma_Z)}
\end{aligned}$$

which apparently acts only on the second qubit.

Problem 3. Create a decoupling scheme for a 3-qubit molecule with the following Hamiltonian:

$$H = \frac{1}{2}(-\omega_A\sigma_Z^A - \omega_B\sigma_Z^B - \omega_C\sigma_Z^C + \Gamma_{AB}\sigma_Z^A \otimes \sigma_Z^B + \Gamma_{BC}\sigma_Z^B \otimes \sigma_Z^C + \Gamma_{AC}\sigma_Z^A \otimes \sigma_Z^C).$$

In other words, find the decoupling operators U_k such that

$$U = \prod_{k=1}^{n} U_k e^{-iHt}$$

operates only on individual qubits. (You might come up with 3 or 4 decoupling operators depending on what you eventually get.)

Solution:

By the same ideas that we learned in the previous problem, we can use $U_1 = U_2 = \sigma_X^A$, to get rid of all terms corresponding to system A:

$$\begin{aligned} W &= \sigma_X^A e^{-iHt} \sigma_X^A e^{-iHt} \\ &= \exp(i\omega_B \sigma_Z^B t + i\omega_C \sigma_Z^C t - i\Gamma_{BC} \sigma_Z^B \otimes \sigma_Z^C t) \end{aligned}$$

and then, using the same trick with W and σ_X^B , we have

$$\begin{aligned} V &= \sigma_X^B W \sigma_X^B W \\ &= \sigma_X^B \sigma_X^A e^{-iHt} \sigma_X^A e^{-iHt} \sigma_X^B \sigma_X^A e^{-iHt} \sigma_X^A e^{-iHt} \\ &= \exp(2i\omega_C \sigma_Z^C t). \end{aligned}$$