MASSACHUSETTS INSTITUTE OF TECHNOLOGY

2.111J/18.435J/ESD.79
Quantum Computation
Fall 2004

Problem Set 4
Solutions

Problem 1. Verify that $|\nearrow\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}|1\rangle$ and $|\swarrow\rangle = -\sin\frac{\theta}{2}|0\rangle + \cos\frac{\theta}{2}|1\rangle$ are the corresponding eigenvectors to, respectively, the eigenvalues +1 and –1 of the operator $\sigma = \cos\theta\sigma_Z + \sin\theta\sigma_X$.

Solution:

$$\begin{aligned}
\sigma|\nearrow\rangle &= (\cos\theta\sigma_Z + \sin\theta\sigma_X)(\cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}|1\rangle) \\
&= (\cos\theta\cos\frac{\theta}{2} + \sin\theta\sin\frac{\theta}{2})|0\rangle + (-\cos\theta\sin\frac{\theta}{2} + \sin\theta\cos\frac{\theta}{2})|1\rangle \\
&= \cos(\theta - \frac{\theta}{2})|0\rangle + \sin(\theta - \frac{\theta}{2})|1\rangle \\
&= |\nearrow\rangle
\end{aligned}$$

$$\begin{aligned}
\sigma|\swarrow\rangle &= (\cos\theta\sigma_Z + \sin\theta\sigma_X)(-\sin\frac{\theta}{2}|0\rangle + \cos\frac{\theta}{2}|1\rangle) \\
&= (-\cos\theta\sin\frac{\theta}{2} + \sin\theta\cos\frac{\theta}{2})|0\rangle + (-\cos\theta\cos\frac{\theta}{2} - \sin\theta\sin\frac{\theta}{2})|1\rangle \\
&= \sin(\theta - \frac{\theta}{2})|0\rangle - \cos(\theta - \frac{\theta}{2})|1\rangle \\
&= -|\swarrow\rangle
\end{aligned}$$

Problem 2. Show that

$$(|00\rangle_{AB} + |11\rangle_{AB})/\sqrt{2} = (|\nearrow\rangle_A|\nearrow\rangle_B + |\swarrow\rangle_A|\swarrow\rangle_B)/\sqrt{2}$$

where $|\nearrow\rangle$ and $|\swarrow\rangle$ are defined in Problem 1.

Solution:

$$\begin{aligned}
|\nearrow\rangle_A|\nearrow\rangle_B &= (\cos\frac{\theta}{2}|0\rangle_A + \sin\frac{\theta}{2}|1\rangle_A) \otimes (\cos\frac{\theta}{2}|0\rangle_B + \sin\frac{\theta}{2}|1\rangle_B) \\
&= \left(\cos\frac{\theta}{2}\right)^2 |0\rangle_A|0\rangle_B + \cos\frac{\theta}{2}\sin\frac{\theta}{2}|0\rangle_A|1\rangle_B
\end{aligned}$$

$$+\sin\frac{\theta}{2}\cos\frac{\theta}{2}|1\rangle_A|0\rangle_B + \left(\sin\frac{\theta}{2}\right)^2 |1\rangle_A|1\rangle_B$$

Similarly,

$$\begin{aligned}|\swarrow\!\!\!\nearrow\rangle_A|\swarrow\!\!\!\nearrow\rangle_B &= (-\sin\frac{\theta}{2}|0\rangle_A + \cos\frac{\theta}{2}|1\rangle_A)\otimes(-\sin\frac{\theta}{2}|0\rangle_B + \cos\frac{\theta}{2}|1\rangle_B)\\ &= \left(\sin\frac{\theta}{2}\right)^2|0\rangle_A|0\rangle_B - \cos\frac{\theta}{2}\sin\frac{\theta}{2}|0\rangle_A|1\rangle_B\\ &\quad -\sin\frac{\theta}{2}\cos\frac{\theta}{2}|1\rangle_A|0\rangle_B + \left(\cos\frac{\theta}{2}\right)^2|1\rangle_A|1\rangle_B\end{aligned}$$

Adding these two completes the proof.

Problem 3. For the state $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |0\rangle_B + |1\rangle_A \otimes |1\rangle_B)$, it can be seen that

a) $$\begin{aligned}\Pr(\uparrow\nearrow) &\equiv \langle\psi|(|\uparrow\rangle_A\langle\uparrow|\otimes|\nearrow\rangle_B\langle\nearrow|)|\psi\rangle\\ &= |\langle\psi|(|\uparrow\rangle_A\otimes|\nearrow\rangle_B)|^2\\ &= |{}_B\langle 0|\nearrow\rangle_B|^2/2\\ &= \frac{1}{2}\left(\cos\frac{\theta}{2}\right)^2 = \frac{3}{8}\end{aligned}$$

where $\theta = \pi/3$ and $|\nearrow\rangle$ is the +1-eigenstate of the operator σ in Problem 1. $|\uparrow\rangle = |0\rangle$ is the +1-eigenstate of σ_Z. Similarly it can be seen that

b) $\Pr(\uparrow\swarrow) \equiv \langle\psi|(|\uparrow\rangle_A\langle\uparrow|\otimes|\swarrow\rangle_B\langle\swarrow|)|\psi\rangle = \frac{1}{2}\left(\sin\frac{\theta}{2}\right)^2 = \frac{1}{8}$

c) $\Pr(\downarrow\nearrow) \equiv \langle\psi|(|\downarrow\rangle_A\langle\downarrow|\otimes|\nearrow\rangle_B\langle\nearrow|)|\psi\rangle = \frac{1}{2}\left(\sin\frac{\theta}{2}\right)^2 = \frac{1}{8}$

d) $\Pr(\downarrow\swarrow) \equiv \langle\psi|(|\downarrow\rangle_A\langle\downarrow|\otimes|\swarrow\rangle_B\langle\swarrow|)|\psi\rangle = \frac{1}{2}\left(\cos\frac{\theta}{2}\right)^2 = 3/8$

where $\theta = \pi/3$ and $|\swarrow\rangle$ is the (-1)-eigenstate of the operator σ in Problem 1. $|\downarrow\rangle = |1\rangle$ is the (-1)-eigenstate of σ_Z.

Problem 4. For the GHZ state

$$|\psi\rangle = (|0\rangle_A|0\rangle_B|0\rangle_C + |1\rangle_A|1\rangle_B|1\rangle_C)/\sqrt{2}$$

evaluate the following expectation values:

$$\left\langle \sigma_X^A \otimes \sigma_Y^B \otimes \sigma_Y^C \right\rangle = -1$$
$$\left\langle \sigma_Y^A \otimes \sigma_X^B \otimes \sigma_Y^C \right\rangle = -1$$
$$\left\langle \sigma_Y^A \otimes \sigma_Y^B \otimes \sigma_X^C \right\rangle = -1$$
$$\left\langle \sigma_X^A \otimes \sigma_X^B \otimes \sigma_X^C \right\rangle = +1$$

Solution:

$$\begin{aligned}\sigma_X^A \otimes \sigma_Y^B \otimes \sigma_Y^C |\psi\rangle &= \sigma_X^A \otimes \sigma_Y^B \otimes \sigma_Y^C (|0\rangle_A|0\rangle_B|0\rangle_C + |1\rangle_A|1\rangle_B|1\rangle_C)/\sqrt{2} \\ &= (|1\rangle_A\, i|1\rangle_B\, i|1\rangle_C + |0\rangle_A(-i)|0\rangle_B(-i)|0\rangle_C)/\sqrt{2} \\ &= -|\psi\rangle\end{aligned}$$

$\Rightarrow$

$$\begin{aligned}\left\langle \sigma_X^A \otimes \sigma_Y^B \otimes \sigma_Y^C \right\rangle &\equiv \left\langle \psi \middle| \sigma_X^A \otimes \sigma_Y^B \otimes \sigma_Y^C \middle| \psi \right\rangle \\ &= -1\end{aligned}$$

$$\begin{aligned}\sigma_Y^A \otimes \sigma_X^B \otimes \sigma_Y^C |\psi\rangle &= \sigma_Y^A \otimes \sigma_X^B \otimes \sigma_Y^C (|0\rangle_A|0\rangle_B|0\rangle_C + |1\rangle_A|1\rangle_B|1\rangle_C)/\sqrt{2} \\ &= (i|1\rangle_A|1\rangle_B\, i|1\rangle_C + (-i)|0\rangle_A|0\rangle_B(-i)|0\rangle_C)/\sqrt{2} \\ &= -|\psi\rangle\end{aligned}$$

$\Rightarrow$

$$\left\langle \sigma_Y^A \otimes \sigma_X^B \otimes \sigma_Y^C \right\rangle = -1$$

Similarly,

$$\sigma_Y^A \otimes \sigma_Y^B \otimes \sigma_X^C |\psi\rangle = -|\psi\rangle \Rightarrow \left\langle \sigma_Y^A \otimes \sigma_Y^B \otimes \sigma_X^C \right\rangle = -1$$

But,

$$\sigma_X^A \otimes \sigma_X^B \otimes \sigma_X^C |\psi\rangle = |\psi\rangle \Rightarrow \left\langle \sigma_X^A \otimes \sigma_X^B \otimes \sigma_X^C \right\rangle = 1\,.$$

$\sigma_X^A \otimes \sigma_Y^B \otimes \sigma_Y^C$ is the operator that corresponds to taking a measurement of σ_X^A on A, σ_Y^B on B, σ_Y^C on C, and multiplying the results, e.g. getting +1 for A, -1 for B, and –1 for C, the result is (+1)(-1)(-1)=+1! Do you notice anything paradoxical in the above results?

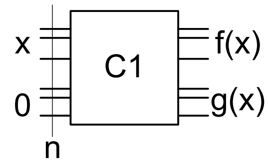
Solution:

From a classical point of view, we can correspond a random variable S_i^j to the result of measurement of σ_i^j, which accepts only values +1 and –1, each with some probability. This is also the case for the product of random variables of this form. For example, $S_Y^A S_X^B S_Y^C$ can only accept values +1 and –1, and therefore, its expected value should fall in [-1,+1]. However, according to our calculations, this expected value is exactly -1, which can only occur iff $S_Y^A S_X^B S_Y^C = -1$ with probability one! Using the same argument, one can obtain

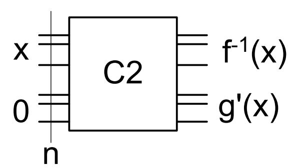
$$
\begin{aligned}
S_X^A \times S_Y^B \times S_Y^C &= -1 \\
S_Y^A \times S_X^B \times S_Y^C &= -1 \\
S_Y^A \times S_Y^B \times S_X^C &= -1 \\
S_X^A \times S_X^B \times S_X^C &= +1
\end{aligned}
$$

The paradox arises from the fact that the total product of the terms on the left hand side of the above equations is a square, and therefore, always nonnegative, but from the values on the right hand side, we get -1!

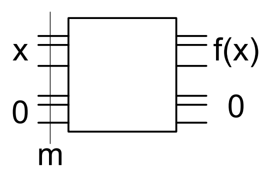
Problem 5. Suppose f is a one-to-one function, which can be constructed using the following circuit:



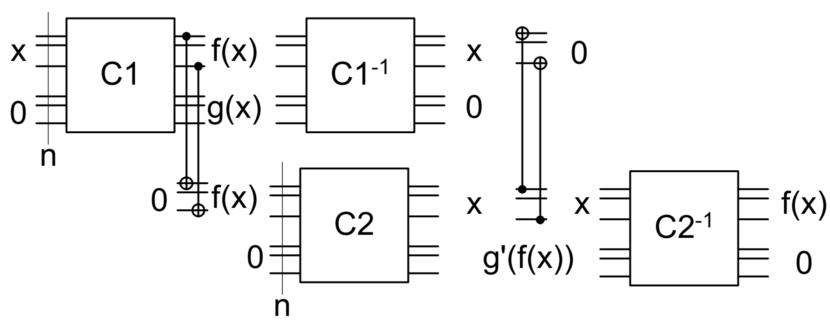
This circuit has C_1 gates and accepts n bits at the input (including data bits *x* and work bits 0). $g(x)$ is the data loaded on the work space after the operation of the circuit. f^{-1} can also be constructed using the following circuit with C_2 gates:



Show that there exists a reversible circuit with the following operation that uses only $k(C_1 + C_2 + n)$ gates where *k* is a small constant:



Solution:



where $C1^{-1}$ and $C2^{-1}$ are the reverse circuits to C1 and C2. The above circuit uses less than $2(C_1 + C_2 + n)$ gates.

Problem 6. Find what CNOT looks like in the basis $\{|++\rangle, |+-\rangle, |-+\rangle, |--\rangle\}$. (write down the corresponding matrix representation.)

Solution:

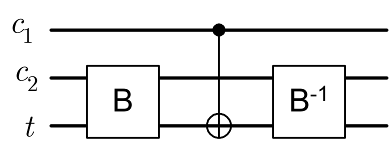
See Exercise 4.20, and equations (4.24)—(4.27). The corresponding matrix looks like this:

$$U_{CNOT} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

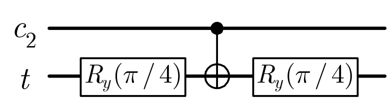
Problem 7. Exercise 4.26 from the Nielsen and Chuang book. Note that the first two rotations, from left, are $R_y(\pi/4)$ and the last two are $R_y(-\pi/4)$ where $R_y(\theta) = \exp(-i\sigma_Y\theta/2)$.

Solution:

The circuit in Exercise 4.26 can be redrawn as follows:



where the block B is as follows:



If $c_1 = 0$, then $BB^{-1} = I$, and the whole circuit is the identity operator. Let's assume $c_1 = 1$. Then,

if $c_2 = 0$, then

$$B = I \otimes (R_y(\pi/4))^2 = I \otimes R_y(\pi/2) = I \otimes (I - i\sigma_Y)/\sqrt{2}$$

if $c_2 = 1$, then

$$B = I \otimes R_y(\pi/4)\sigma_X R_y(\pi/4)$$

where using $\alpha \triangleq \cos(\pi/8)$ and $\beta \triangleq \sin(\pi/8)$, and $\sigma_Y \sigma_X \sigma_Y = i\sigma_Y \sigma_Z = -\sigma_X$

$$\begin{aligned} R_y(\pi/4)\sigma_X R_y(\pi/4) &= (\alpha I - i\beta\sigma_Y)\sigma_X(\alpha I - i\beta\sigma_Y) \\ &= \alpha^2\sigma_X - i\alpha\beta\{\sigma_Y, \sigma_X\} - \beta^2\sigma_Y\sigma_X\sigma_Y \\ &= (\alpha^2 + \beta^2)\sigma_X = \sigma_X, \end{aligned}$$

we have

$$B = I \otimes \sigma_X,\ c_2 = 1$$

Using the above relations, one can obtain

$$\begin{aligned} B|0\rangle|0\rangle &= |0\rangle|+\rangle \\ B|0\rangle|1\rangle &= |0\rangle|-\rangle \\ B|1\rangle|0\rangle &= |1\rangle|1\rangle \\ B|1\rangle|1\rangle &= |1\rangle|0\rangle \end{aligned}$$

The CNOT gate between the first control bit and the test bit, changes $|1\rangle$ to $|0\rangle$ and vice verca, and therefore, it is easy to verify that the first circuit behaves like a Toffoli gate except for the input $|1\rangle_{c_1}|0\rangle_{c_2}|1\rangle_t$ for which there exist an extra factor –1.

Problem 8. Using a bit-query black box, which acts as follows

$$|X\rangle \otimes |b\rangle \longrightarrow |X\rangle \otimes |b \oplus f(X)\rangle$$

make a phase-query black box with the following operation:

$$|X\rangle \longrightarrow (-1)^{f(X)}|X\rangle$$

where $X = b_1 b_2 \ldots b_n$ in the binary representation and $|X\rangle = |b_1\rangle|b_2\rangle\cdots|b_n\rangle$.

Solution:

See Section 6.1.1, equations (6.1)–(6.3).