MASSACHUSETTS INSTITUTE OF TECHNOLOGY

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Quantum Computation
Fall 2004

Problem Set 3
Solution

Problem 1. For a composite system A and B, define

$$\mathrm{I}_{AB}^2 = (\mathrm{I}_X^{AB})^2 + (\mathrm{I}_Y^{AB})^2 + (\mathrm{I}_Z^{AB})^2$$

where

$$\mathrm{I}_j^{AB} = \frac{1}{2}\sigma_j^A \otimes I_B + \frac{1}{2} I_A \otimes \sigma_j^B \text{ for } j \in \{X, Y, Z\}.$$

Evaluate

$$\left\langle \mathrm{I}_{AB}^2 \right\rangle = {}_{AB}\left\langle \psi \middle| \mathrm{I}_{AB}^2 \middle| \psi \right\rangle_{AB}$$

for the following states:

i) $|\psi\rangle_{AB} = (|01\rangle_{AB} - |10\rangle_{AB})/\sqrt{2}$
ii) $|\psi\rangle_{AB} = (|01\rangle_{AB} + |10\rangle_{AB})/\sqrt{2}$
iii) $|\psi\rangle_{AB} = (|00\rangle_{AB} + |11\rangle_{AB})/\sqrt{2}$
iv) $|\psi\rangle_{AB} = (|00\rangle_{AB} - |11\rangle_{AB})/\sqrt{2}$.

Solution:

From the lecture, you should remember that

$$(\mathrm{I}_j^{AB})^2 = \frac{1}{2} I_A \otimes I_B + \frac{1}{2}\sigma_j^A \otimes \sigma_j^B \text{ for } j \in \{X, Y, Z\}.$$

Furthermore,

$$\sigma_X|0\rangle = |1\rangle,\ \sigma_X|1\rangle = |0\rangle,\ \sigma_Y|0\rangle = i|1\rangle,\ \sigma_Y|1\rangle = -i|0\rangle,\ \sigma_Z|0\rangle = |0\rangle,\ \sigma_Z|1\rangle = -|1\rangle$$

Now, it is easy to verify that

i) $(\sigma_j^A \otimes \sigma_j^B)|\psi\rangle_{AB} = -|\psi\rangle_{AB}$ for $j \in \{X, Y, Z\}$

$\Rightarrow$ ${}_{AB}\left\langle \psi \middle| \sigma_j^A \otimes \sigma_j^B \middle| \psi \right\rangle_{AB} = -1$ and ${}_{AB}\left\langle \psi \middle| I_A \otimes I_B \middle| \psi \right\rangle_{AB} = 1$

$\Rightarrow$ ${}_{AB}\left\langle \psi \middle| \left(\mathrm{I}_j^{AB}\right)^2 \middle| \psi \right\rangle_{AB} = 0$ for $j \in \{X, Y, Z\}$

$\Rightarrow$ ${}_{AB}\left\langle \psi \middle| \mathrm{I}_{AB}^2 \middle| \psi \right\rangle_{AB} = 0$

ii) $(\sigma_X^A \otimes \sigma_X^B)|\psi\rangle_{AB} = (\sigma_Y^A \otimes \sigma_Y^B)|\psi\rangle_{AB} = -(\sigma_Z^A \otimes \sigma_Z^B)|\psi\rangle_{AB} = |\psi\rangle_{AB}$

$\Rightarrow \quad {}_{AB}\big\langle\psi\big|\mathrm{I}_{AB}^2\big|\psi\big\rangle_{AB} = 2\,.$

iii) $(\sigma_X^A \otimes \sigma_X^B)|\psi\rangle_{AB} = -(\sigma_Y^A \otimes \sigma_Y^B)|\psi\rangle_{AB} = (\sigma_Z^A \otimes \sigma_Z^B)|\psi\rangle_{AB} = |\psi\rangle_{AB}$
$\Rightarrow \quad {}_{AB}\big\langle\psi\big|\mathrm{I}_{AB}^2\big|\psi\big\rangle_{AB} = 2\,.$

iv) $-(\sigma_X^A \otimes \sigma_X^B)|\psi\rangle_{AB} = (\sigma_Y^A \otimes \sigma_Y^B)|\psi\rangle_{AB} = (\sigma_Z^A \otimes \sigma_Z^B)|\psi\rangle_{AB} = |\psi\rangle_{AB}$
$\Rightarrow \quad {}_{AB}\big\langle\psi\big|\mathrm{I}_{AB}^2\big|\psi\big\rangle_{AB} = 2\,.$

Problem 2. Recall

$$\begin{aligned}[\sigma_X, \sigma_Y] &\equiv \sigma_X\sigma_Y - \sigma_Y\sigma_X \\ &= 2i\sigma_Z\,.\end{aligned}$$

Find the following commutation relations:

$[\sigma_X^A \otimes \sigma_X^B, \sigma_Y^A \otimes \sigma_Y^B]$, $[\sigma_Y^A \otimes \sigma_Y^B, \sigma_Z^A \otimes \sigma_Z^B]$, and $[\sigma_Z^A \otimes \sigma_Z^B, \sigma_X^A \otimes \sigma_X^B]$.

Solution:

$$\begin{aligned}[\sigma_X^A \otimes \sigma_X^B, \sigma_Y^A \otimes \sigma_Y^B] &= (\sigma_X^A \otimes \sigma_X^B)(\sigma_Y^A \otimes \sigma_Y^B) - (\sigma_Y^A \otimes \sigma_Y^B)(\sigma_X^A \otimes \sigma_X^B) \\ &= \sigma_X^A\sigma_Y^A \otimes \sigma_X^B\sigma_Y^B - \sigma_Y^A\sigma_X^A \otimes \sigma_Y^B\sigma_X^B \\ &= i\sigma_Z^A \otimes i\sigma_Z^B - (-i)\sigma_Z^A \otimes (-i)\sigma_Z^B \\ &= 0\,.\end{aligned}$$

Other commutation relations are also zero using a similar treatment.

Problem 3. For the state $|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |1\rangle_B - |1\rangle_A \otimes |0\rangle_B)$, find

$$\rho_A \equiv tr_B(\rho_{AB})$$

where

$$\rho_{AB} = |\psi\rangle_{AB}\langle\psi|\,.$$

Solution:

$$\begin{aligned}\rho_A &\equiv tr_B(\rho_{AB}) \\ &\equiv_B \big\langle 0\big|\rho_{AB}\big|0\big\rangle_B + {}_B\big\langle 1\big|\rho_{AB}\big|1\big\rangle_B \\ &= {}_B\langle 0|\psi\rangle_{AB}\langle\psi|0\rangle_B + {}_B\langle 1|\psi\rangle_{AB}\langle\psi|1\rangle_B \\ &= \frac{1}{2}(|1\rangle_{AA}\langle 1| + |0\rangle_{AA}\langle 0|)\end{aligned}$$

Problem 4. A C-NOT gate can be represented by the following unitary operator:

$$U_{CNOT} = |0\rangle_A \langle 0| \otimes I_B + |1\rangle_A \langle 1| \otimes \sigma_X^B .$$

Verify that $U_{CNOT} = U_{CNOT}^\dagger$ and $U_{CNOT}^2 = I$.

Solution:

Projectors, the identity operator, and the Pauli matrices are all Hermitian. Therefore, $U_{CNOT} = U_{CNOT}^\dagger$.

$$\begin{aligned}
U_{CNOT}^2 &= \left(|0\rangle_A \langle 0| \otimes I_B + |1\rangle_A \langle 1| \otimes \sigma_X^B\right)\left(|0\rangle_A \langle 0| \otimes I_B + |1\rangle_A \langle 1| \otimes \sigma_X^B\right) \\
&= |0\rangle_A \langle 0|0\rangle_A \langle 0| \otimes I_B^2 + |0\rangle_A \langle 0|1\rangle_A \langle 1| \otimes I_B \sigma_X^B \\
&\quad + |1\rangle_A \langle 1|0\rangle_A \langle 0| \otimes \sigma_X^B I_B + |1\rangle_A \langle 1|1\rangle_A \langle 1| \otimes \sigma_X^B \sigma_X^B \\
&= |0\rangle_A \langle 0| \otimes I_B + |1\rangle_A \langle 1| \otimes \left(\sigma_X^B\right)^2 \\
&= \left(|0\rangle_A \langle 0| + |1\rangle_A \langle 1|\right) \otimes I_B \\
&= I_A \otimes I_B .
\end{aligned}$$

Problem 5. Suppose your systems have 3-D vector spaces with $\{|0\rangle, |1\rangle, |2\rangle\}$ as the basis. For the operators

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{bmatrix}, \ \omega = e^{2\pi i/3} \quad (R|j\rangle = e^{2\pi i j/3}|j\rangle)$$

$$T = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad T|j\rangle = |(j+1) \bmod 3\rangle$$

a) Find the commutation relation among *R* and *T*, $[R, T]$.

b) Show how Alice and Bob can start with $|\psi\rangle = (|00\rangle + |11\rangle + |22\rangle)/\sqrt{3}$ and use operators $R^a T^b$, for *a* and *b* integers, to send two classical trits (9 different messages) using one qutrit of communication.

Solution:

a)

$$RT = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{bmatrix} \times \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ \omega & 0 & 0 \\ 0 & \omega^2 & 0 \end{bmatrix}$$

$$TR = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \omega^2 \\ 1 & 0 & 0 \\ 0 & \omega & 0 \end{bmatrix} = \omega^2 RT$$

$$\Rightarrow \quad RT = \omega TR$$
$$\Rightarrow \quad R^a T = \omega^a T R^a$$
$$\Rightarrow \quad R^a T^b = \omega^{ab} T^b R^a$$

for a and b integers (the above proof is for positive integers, but it is easy to extend it to all integers). In particular,

$$[R,T] = (1-\omega^2)RT \,.$$

Also note that

$$R^\dagger = R^{-1}, T^\dagger = T^{-1}$$
$$R^3 = I, T^3 = I \,.$$

b) This is a generalization of the superdense coding to the 3-D case. Alice chooses a pair $\{a,b\}$, $0 \le a,b \le 2$, and then uses the operator $R^a T^b$ to encode her qutrit. Then, she sends her qutrit to Bob via a quantum channel. In order to make it possible for Bob to distinguish between all nine possible operations that Alice can perform, we have to show that the states $R^a T^b |\psi\rangle$ are pair-wise orthogonal for different pairs $\{a,b\}$. In fact, we have to calculate the inner product of $R^a T^b |\psi\rangle$ and $R^c T^d |\psi\rangle$ for different pairs $\{a,b\}$ and $\{c,d\}$, which is as follows:

$$\begin{aligned} \left\langle \psi \middle| (R^c\, T^d)^\dagger R^a T^b \middle| \psi \right\rangle &= \left\langle \psi \middle| T^{-d} R^{a-c} T^b \middle| \psi \right\rangle \\ &= \omega^{b(a-c)} \left\langle \psi \middle| T^{b-d} R^{a-c} \middle| \psi \right\rangle \\ &= \frac{\omega^{b(a-c)}}{\sqrt{3}} \langle \psi \big| T^{b-d} \left(|00\rangle + \omega^{a-c} |11\rangle + \omega^{2(a-c)} |22\rangle \right) \end{aligned}$$
$$= \frac{\omega^{b(a-c)}}{\sqrt{3}} \langle \psi \big| \left(|b-d\rangle_A |0\rangle_B + \omega^{a-c} |b-d+1\rangle_A |1\rangle_B + \omega^{2(a-c)} |b-d+2\rangle_A |2\rangle_B \right)$$
$$\equiv A$$

For $b = d$ and $a = c$, we have $A = 1$, which proves that the vectors are normal.
For $b = d$ and $a \ne c$, we have $A = 1 + \omega^{a-c} + \omega^{2(a-c)} = 1 + e^{2\pi i/3} + e^{4\pi i/3} = 0$. Similarly, it is easy to verify that for the case of $b \ne d$, all vectors on the right hand side are orthogonal to the ones on the left; therefore, $A = 0$. This proves that the vectors $R^a T^b |\psi\rangle$ are orthonormal. So, there is a unitary transform by which Bob can transform these set of vectors to the computational basis, and then he can easily distinguish between different states by doing a measurement in the computational basis.