MASSACHUSETTS INSTITUTE OF TECHNOLOGY

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Quantum Computation
Fall 2004

Problem Set 9
Solutions

Problem 1. For the position operator x and the momentum operator p (corresponding to $-i\hbar\frac{\partial}{\partial x}$), show that $[x,p]=i\hbar$.

Solution:

For any function $f(x)$, we have

$$\begin{aligned}[x,p]f(x) &= xpf(x) - pxf(x)\\ &= x\frac{\partial}{\partial x}f(x) - \frac{\partial}{\partial x}\big(xf(x)\big)\\ &= -i\hbar x\frac{\partial}{\partial x}f(x) + i\hbar x\frac{\partial}{\partial x}\big(f(x)\big) + i\hbar f(x)\\ &= i\hbar f(x)\end{aligned}$$

$\Rightarrow$

$$[x,p]=i\hbar\,.$$

Problem 2. Verify the following equalities:
(a) $[a,a^\dagger]=1$
(b) $[a,a^\dagger a]=a$
(c) $[a^\dagger,a^\dagger a]=-a^\dagger$
(d) $H=\hbar\omega(a^\dagger a+1/2)$
where

$$a=\frac{1}{\sqrt{2\hbar\omega}}(\omega x+ip)$$

and

$$H=\frac{1}{2}(p^2+\omega^2x^2)\,.$$

Solution:

(a)
$$\begin{aligned}[a,a^\dagger] &= \frac{1}{2\hbar\omega}[\omega x+ip,\omega x-ip]\\ &= \frac{1}{2\hbar\omega}\big(\omega^2[x,x]+[p,p]-i\omega[x,p]+i\omega[p,x]\big)\end{aligned}$$

$$= \frac{1}{2\hbar\omega}(-i\omega(i\hbar) + i\omega(-i\hbar))$$
$$= 1\,.$$

(b) $[a, a^\dagger a] = aa^\dagger a - a^\dagger aa$
$$= (aa^\dagger - a^\dagger a)a$$
$$= [a, a^\dagger]a$$
$$= a\,.$$

(c) $[a^\dagger, a^\dagger a] = a^\dagger(a^\dagger a - aa^\dagger)$
$$= -a^\dagger\,.$$

(d) $\hbar\omega(a^\dagger a + 1/2) = \hbar\omega\left[\frac{1}{2\hbar\omega}(\omega x - ip)(\omega x + ip) + \frac{1}{2}\right]$
$$= \big(\omega^2 x^2 + p^2 + i\omega[x,p] + \hbar\omega\big)/2$$
$$= \frac{1}{2}(p^2 + \omega^2 x^2)$$
$$= H\,.$$

Problem 3. For $\hbar = \omega = 1$, verify $a|0\rangle = 0$, where

$$|0\rangle = \int e^{-x^2/2}\,|x\rangle\,dx\,.$$

Solution:

$$a|0\rangle = \frac{1}{\sqrt{2\hbar\omega}}(\omega x + ip)\int e^{-x^2/2}\,|x\rangle\,dx$$
$$= \frac{1}{\sqrt{2\hbar\omega}}\int \left(\omega x e^{-x^2/2} + i(-i\hbar)\frac{\partial}{\partial x}e^{-x^2/2}\right)|x\rangle\,dx$$
$$= \frac{1}{\sqrt{2\hbar\omega}}\int (\omega - \hbar\,)xe^{-x^2/2}\,|x\rangle\,dx$$
$$= 0$$

Problem 4. Verify

$$a|n\rangle = \sqrt{n}\,|n-1\rangle$$

and

$$a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

where

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle\,.$$

Also, show that $|n\rangle$ is an eigenstate of H with energy (eigenvalue) $\hbar\omega(n + 1/2)$.

Solution:

$$
\begin{aligned}
a^{\dagger}|n\rangle &= a^{\dagger}\frac{(a^{\dagger})^{n}}{\sqrt{n!}}\ \ 0\rangle \\
&= \sqrt{n+1}\,\frac{(a^{\dagger})^{n+1}}{\sqrt{(n+1)!}}|0\rangle \\
&= \sqrt{n+1}\,|n+1\rangle .
\end{aligned}
$$

$$
\begin{aligned}
a\ \ n\rangle &= a\frac{(a^{\dagger})^{n}}{\sqrt{n!}}\ \ 0\rangle \\
&= aa^{\dagger}\frac{(a^{\dagger})^{n-1}}{\sqrt{n!}}\ \ 0\rangle \\
&= (1+a^{\dagger}a)\frac{1}{\sqrt{n}}\frac{(a^{\dagger})^{n-1}}{\sqrt{(n-1)!}}\ \ 0\rangle \\
&= (\ \ n-1\rangle + a^{\dagger}a\ \ n-1\rangle)/\sqrt{n}
\end{aligned}
$$

but, by induction, we have

$$
a^{\dagger}a\ \ n-1\rangle = \sqrt{n-1}a^{\dagger}\ \ n-2\rangle = (n-1)\ \ n-1\rangle
$$

$\Rightarrow$

$$
a\ \ n\rangle = \sqrt{n}\ \ n-1\rangle .
$$

Similarly,

$$
\begin{aligned}
H|n\rangle &= \hbar\omega(a^{\dagger}a+1/2)|n\rangle \\
&= \hbar\omega(n+1/2)|n\rangle .
\end{aligned}
$$

Problem 5. Show that $[C,D]=[C,H]=[D,H]=0$, where

$$
C = \omega(a^{\dagger}a+\sigma_{Z}/2),
$$

$$
D = \kappa(a^{\dagger}\sigma_{-}+a\sigma_{+}) - (\Delta\omega/2)\sigma_{Z},
$$

and

$$
H = \hbar(C+D+\omega/2).
$$

(Remember that $\sigma_{\pm} = (\sigma_X \pm i\sigma_Y)/2$, or equivalently $\sigma_{+} = |0\rangle\langle 1|$ and $\sigma_{-} = |1\rangle\langle 0|$.)

Solution:

We have

$$
[a^{\dagger}a,\sigma_{Z}] = 0
$$

$$
\sigma_{Z}\sigma_{+} = (|0\rangle\langle 0| - |1\rangle\langle 1|)|0\rangle\langle 1| = |0\rangle\langle 1| = \sigma_{+}
$$

$$
\sigma_{+}\sigma_{Z} = |0\rangle\langle 1|(|0\rangle\langle 0| - |1\rangle\langle 1|) = -|0\rangle\langle 1| = -\sigma_{+}
$$

$$\sigma_-\sigma_Z = |1\rangle\langle 0|\,(|0\rangle\langle 0| - |1\rangle\langle 1|) = |1\rangle\langle 0| = \sigma_-$$
$$\sigma_Z\sigma_- = (|0\rangle\langle 0| - |1\rangle\langle 1|)\,|1\rangle\langle 0| = -|1\rangle\langle 0| = -\sigma_-$$

$\Rightarrow$

$$\begin{aligned}
[C,D] &= [\omega(a^\dagger a + \sigma_Z/2), \kappa(a^\dagger\sigma_- + a\sigma_+) - (\Delta\omega/2)\sigma_Z] \\
&= \omega\kappa[a^\dagger a, a^\dagger]\otimes\sigma_- + \omega\kappa[a^\dagger a, a]\otimes\sigma_+ + \frac{\omega\kappa}{2}a^\dagger\otimes[\sigma_Z,\sigma_-] + \frac{\omega\kappa}{2}a\otimes[\sigma_Z,\sigma_+] \\
&= \omega\kappa a^\dagger\otimes\sigma_- - \omega\kappa a\otimes\sigma_+ - \omega\kappa a^\dagger\otimes\sigma_- + \omega\kappa a\otimes\sigma_+ \\
&= 0.
\end{aligned}$$

$$[C,H] = [C, \hbar(C + D + \omega/2)] = \hbar\left([C,C] + [C,D] + [C,\omega I/2]\right) = 0.$$
$$[D,H] = [D, \hbar(C + D + \omega/2)] = \hbar\left([D,C] + [D,D] + [D,\omega I/2]\right) = 0.$$

Problem 6. Find the cluster state for a square. In other words, find the state $|\psi\rangle$ such that

$$\sigma_X^A\sigma_Z^B\sigma_Z^D|\psi\rangle = \sigma_X^B\sigma_Z^C\sigma_Z^A|\psi\rangle = \sigma_X^C\sigma_Z^D\sigma_Z^B|\psi\rangle = \sigma_X^D\sigma_Z^A\sigma_Z^C|\psi\rangle = |\psi\rangle$$

where the super indices A, B, C, and D are associated with the vertices of the square.

Solution:

A general eigenstate $|\psi\rangle$ with eigenvalue 1 for $\sigma_X^A\sigma_Z^B\sigma_Z^D$ is as follows:

$$\begin{aligned}
|\psi\rangle = {} & \alpha_1|+\rangle_A|0\rangle_B|0\rangle_C|0\rangle_D + \alpha_2|+\rangle_A|0\rangle_B|1\rangle_C|0\rangle_D + \\
& \alpha_3|+\rangle_A|1\rangle_B|0\rangle_C|1\rangle_D + \alpha_4|+\rangle_A|1\rangle_B|1\rangle_C|1\rangle_D + \\
& \alpha_5|-\rangle_A|0\rangle_B|0\rangle_C|1\rangle_D + \alpha_6|-\rangle_A|0\rangle_B|1\rangle_C|1\rangle_D + \\
& \alpha_7|-\rangle_A|1\rangle_B|0\rangle_C|0\rangle_D + \alpha_8|-\rangle_A|1\rangle_B|1\rangle_C|0\rangle_D
\end{aligned}$$

where $\sum_{k=1}^{8}|\alpha_k|^2 = 1$. In order to satisfy the second equation stated in the problem, we have

$$\begin{aligned}
\sigma_Z^A\sigma_X^B\sigma_Z^C|\psi\rangle = {} & \alpha_1|-\rangle_A|1\rangle_B|0\rangle_C|0\rangle_D - \alpha_2|-\rangle_A|1\rangle_B|1\rangle_C|0\rangle_D + \\
& \alpha_3|-\rangle_A|0\rangle_B|0\rangle_C|1\rangle_D - \alpha_4|-\rangle_A|0\rangle_B|1\rangle_C|1\rangle_D + \\
& \alpha_5|+\rangle_A|1\rangle_B|0\rangle_C|1\rangle_D - \alpha_6|+\rangle_A|1\rangle_B|1\rangle_C|1\rangle_D + \\
& \alpha_7|+\rangle_A|0\rangle_B|0\rangle_C|0\rangle_D - \alpha_8|+\rangle_A|0\rangle_B|1\rangle_C|0\rangle_D \\
= {} & |\psi\rangle
\end{aligned}$$

$\Rightarrow$

$$\alpha_1 = \alpha_7, \alpha_3 = \alpha_5, \alpha_2 = -\alpha_8, \alpha_4 = -\alpha_6.$$

Similarly, by applying the other two operators we get:

$$\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = -\alpha_6 = \alpha_7 = -\alpha_8,$$

which after normalization results in the following cluster state:

$$\begin{aligned} |\psi\rangle = {} & \frac{1}{2}|+\rangle_A|0\rangle_B|+\rangle_C|0\rangle_D + \frac{1}{2}|+\rangle_A|1\rangle_B|+\rangle_C|1\rangle_D \\ & + \frac{1}{2}|-\rangle_A|0\rangle_B|-\rangle_C|1\rangle_D + \frac{1}{2}|-\rangle_A|1\rangle_B|-\rangle_C|0\rangle_D . \end{aligned}$$