MASSACHUSETTS INSTITUTE OF TECHNOLOGY

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Quantum Computation
Fall 2004

Problem Set 7
Solutions

Problem 1. Verify Trotter formula:

$$\left(e^{-iBt/n}e^{-iAt/n}\right)^n = e^{-i(A+B)t} + O(1/n)$$

where A and B are Hermitian operators.

Solution:

See Theorem 4.3, eqn. (4.98), page 207 of N&C.

Problem 2. Using the fact that if we can perform Hermitian operators A and B, then we can perform $\pm i[A,B]$ as well, show how we can get operators $\sigma_i \otimes \sigma_j$ from the set of operators $\sigma_Z \otimes \sigma_Z$, $\sigma_i \otimes I$, and $I \otimes \sigma_j$, where $i,j \in \{X,Y,Z\}$.

Solution:

We have

$$i[\sigma_m \otimes \sigma_n, \sigma_k \otimes I] = -2[\sigma_m, \sigma_k] \otimes \sigma_n \tag{1}$$

and

$$i[\sigma_m \otimes \sigma_n, I \otimes \sigma_k] = -2\sigma_m \otimes [\sigma_n, \sigma_k] \tag{2}$$

Assuming $\sigma_m \otimes \sigma_n = \sigma_Z \otimes \sigma_Z$ in (1), one can obtain, by appropriate choice of k, operators in the form $\sigma_i \otimes \sigma_Z$, for any i. Now, assuming $\sigma_m \otimes \sigma_n = \sigma_i \otimes \sigma_Z$ in (2), one can obtain, again by changing k, operators in the form $\sigma_i \otimes \sigma_j$, for any i and j.

Problem 3. Use the same trick to show how to produce an arbitrary operator

$$\sigma_{i_1} \otimes \sigma_{i_2} \otimes \cdots \otimes \sigma_{i_n}.$$

Solution:

We can construct the above operator inductively. The case of $n = 2$ was shown in Problem 2. Now, assuming we can construct any operator in the form

$\sigma_{i_1} \otimes \sigma_{i_2} \otimes \cdots \otimes \sigma_{i_{n-1}}$, or equivalently, $\sigma_{i_1} \otimes \sigma_{i_2} \otimes \cdots \otimes \sigma_{i_{n-1}} \otimes I$, one can use eqn (2) of problem 2 to construct $\sigma_{i_1} \otimes \sigma_{i_2} \otimes \cdots \otimes \sigma_{i_n}$.

Problem 4.
(a) Construct a CNOT gate using only $e^{-i\phi\sigma_Z \otimes \sigma_Z}$ and $e^{-i\theta\sigma}$ gates.
(b) What is the smallest number of one-qubit gates and the two-qubit gate $\sigma_Z \otimes \sigma_Z$ needed for the construction of a CNOT gate.

Solution:

In exercise 4.17, it has been shown that we can build a CNOT gate using a controlled-Z gate with the following matrix representation:

$$U_{CZ} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

and two Hadamard gates. In other words,

$$U_{CNOT} = (I \otimes H) U_{CZ} (I \otimes H).$$

The above matrix is a diagonal matrix, and in order to construct it, up to an overall phase, we just need to play with rotations $\exp(-i\phi\sigma_Z \otimes \sigma_Z)$, $\exp(-i\gamma\sigma_Z \otimes I)$, and $\exp(-i\theta I \otimes \sigma_Z)$. A simple verification shows that for $\phi = -\pi/4$ and $\theta = \gamma = \pi/4$, one can obtain the above matrix up to an overall phase $\exp(-i\pi/4)$. To complete the CNOT construction, we need to implement the Hadamard gate using single-qubit rotations. It can be done, as shown in Problem 1 of the midterm, using a π-radian rotation about $(\hat{x} + \hat{z})/\sqrt{2}$. Therefore, one solution to this problem is

$$e^{-i\pi/4} U_{CNOT} = e^{-i\frac{\pi}{2} I \otimes \frac{\sigma_X + \sigma_Z}{\sqrt{2}}} e^{i\frac{\pi}{4}\sigma_Z \otimes \sigma_Z} e^{-i\frac{\pi}{4}\sigma_Z \otimes I} e^{-i\frac{\pi}{4} I \otimes \sigma_Z} e^{i\frac{\pi}{2} I \otimes \frac{\sigma_X + \sigma_Z}{\sqrt{2}}}.$$

Problem 5. Verify

(a) $\cos(\omega t)\sigma_X - \sin(\omega t)\sigma_Y = e^{i\omega t}\sigma_+ + e^{-i\omega t}\sigma_-$
(b) $e^{\pm i\omega t}\sigma_\pm e^{i\omega t\sigma_Z/2} = e^{i\omega t\sigma_Z/2}\sigma_\pm$

where $\sigma_\pm = (\sigma_X \pm i\sigma_Y)/2$.

Solution:

(a) $$\begin{aligned} e^{i\omega t}\sigma_+ + e^{-i\omega t}\sigma_- &= e^{i\omega t}(\sigma_X + i\sigma_Y)/2 + e^{-i\omega t}(\sigma_X - i\sigma_Y)/2 \\ &= (e^{i\omega t} + e^{-i\omega t})\sigma_X/2 + i(e^{i\omega t} - e^{-i\omega t})\sigma_Y/2 \\ &= \cos(\omega t)\sigma_X - \sin(\omega t)\sigma_Y. \end{aligned}$$

(b) $$\begin{aligned} e^{\pm i\omega t}\sigma_\pm e^{i\omega t\sigma_Z/2} &= e^{\pm i\omega t}\sigma_\pm(\cos(\omega t/2) + i\sin(\omega t/2)\sigma_Z) \\ &= e^{\pm i\omega t}(\sigma_\pm\cos(\omega t/2) + i\sin(\omega t/2)\sigma_\pm\sigma_Z) \\ &= e^{\pm i\omega t}(\frac{\sigma_X \pm i\sigma_Y}{2}\cos(\omega t/2) + i\sin(\omega t/2)\frac{\sigma_X \pm i\sigma_Y}{2}\sigma_Z) \\ &= e^{\pm i\omega t}(\frac{\sigma_X \pm i\sigma_Y}{2}\cos(\omega t/2) - i\sin(\omega t/2)\frac{i\sigma_Y \pm \sigma_X}{2}) \\ &= e^{\pm i\omega t}(\cos(\omega t/2) \mp i\sin(\omega t/2))\sigma_\pm \\ &= e^{\pm i\omega t/2}\sigma_\pm. \end{aligned}$$

On the other hand,

$$\begin{aligned} e^{i\omega t\sigma_Z/2}\sigma_\pm &= (\cos(\omega t/2) + i\sin(\omega t/2)\sigma_Z)\frac{\sigma_X \pm i\sigma_Y}{2} \\ &= (\cos(\omega t/2)\frac{\sigma_X \pm i\sigma_Y}{2} + i\sin(\omega t/2)\sigma_Z\frac{\sigma_X \pm i\sigma_Y}{2}) \\ &= (\cos(\omega t/2)\sigma_\pm + i\sin(\omega t/2)\frac{i\sigma_Y \pm \sigma_X}{2}) \\ &= e^{\pm i\omega t/2}\sigma_\pm. \end{aligned}$$

$\Rightarrow$

$$e^{\pm i\omega t}\sigma_\pm e^{i\omega t\sigma_Z/2} = e^{i\omega t\sigma_Z/2}\sigma_\pm.$$