# MASSACHUSETTS INSTITUTE OF TECHNOLOGY

2.111J/18.435J/ESD.79
Quantum Computation
Fall 2004

**Problem Set 4**
Due: Tuesday, October 12 (in class)

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**Problem 1.** Verify that $|\nearrow\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}|1\rangle$ and $|\swarrow\rangle = -\sin\frac{\theta}{2}|0\rangle + \cos\frac{\theta}{2}|1\rangle$ are the corresponding eigenvectors to, respectively, the eigenvalues +1 and –1 of the operator $\sigma = \cos\theta\sigma_Z + \sin\theta\sigma_X$.

**Problem 2.** Show that

$$(|00\rangle_{AB} + |11\rangle_{AB})/\sqrt{2} = (|\nearrow\rangle_A|\nearrow\rangle_B + |\swarrow\rangle_A|\swarrow\rangle_B)/\sqrt{2}$$

where $|\nearrow\rangle$ and $|\swarrow\rangle$ are defined in Problem 1.

**Problem 3.** For the state $|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |0\rangle_B + |1\rangle_A \otimes |1\rangle_B)$, verify that

**a)** $\Pr(\uparrow\nearrow) \equiv tr(|\psi\rangle_{AB}\langle\psi||\uparrow\rangle_A\langle\uparrow| \otimes |\nearrow\rangle_B\langle\nearrow|)$
$= \frac{1}{2}\left(\cos\frac{\theta}{2}\right)^2 = \frac{3}{8}$

where $\theta = \pi/3$ and $|\nearrow\rangle$ is the +1-eigenstate of the operator σ in Problem 1. $|\uparrow\rangle = |0\rangle$ is the +1-eigenstate of $\sigma_Z$. Similarly verify that

**b)** $\Pr(\uparrow\swarrow) = \frac{1}{2}\left(\sin\frac{\theta}{2}\right)^2 = \frac{1}{8}$
**c)** $\Pr(\downarrow\nearrow) = 1/8$
**d)** $\Pr(\downarrow\swarrow) = 3/8$

where $\theta = \pi/3$ and $|\swarrow\rangle$ is the (-1)-eigenstate of the operator σ in Problem 1. $|\downarrow\rangle = |1\rangle$ is the (-1)-eigenstate of $\sigma_Z$.

**Problem 4.** For the GHZ state

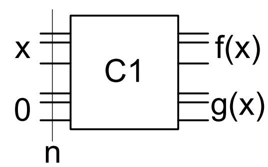
$$|\psi\rangle = (|0\rangle_A|0\rangle_B|0\rangle_C + |1\rangle_A|1\rangle_B|1\rangle_C)/\sqrt{2}$$

evaluate the following expectation values:

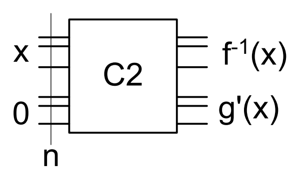
$$\left\langle \sigma_X^A \otimes \sigma_Y^B \otimes \sigma_Y^C \right\rangle = ?$$
$$\left\langle \sigma_Y^A \otimes \sigma_X^B \otimes \sigma_Y^C \right\rangle = ?$$
$$\left\langle \sigma_Y^A \otimes \sigma_Y^B \otimes \sigma_X^C \right\rangle = ?$$
$$\left\langle \sigma_X^A \otimes \sigma_X^B \otimes \sigma_X^C \right\rangle = ?$$

$\sigma_X^A \otimes \sigma_Y^B \otimes \sigma_Y^C$ is the operator that corresponds to taking a measurement of $\sigma_X^A$ on A, $\sigma_Y^B$ on B, $\sigma_Y^C$ on C, and multiplying the results, e.g. getting +1 for A, -1 for B, and –1 for C, the result is (+1)(-1)(-1)=+1! Do you notice anything paradoxical in the above results?

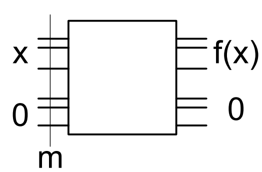
**Problem 5.** Suppose $f$ is a one-to-one function, which can be constructed using the following circuit:




This circuit has $C_1$ gates and accepts $n$ bits at the input (including data bits *x* and work bits 0). $g(x)$ is the data loaded on the work space after the operation of the circuit. $f^{-1}$ can also be constructed using the following circuit with $C_2$ gates:




Show that there exists a reversible circuit with the following operation that uses only $k(C_1 + C_2 + n)$ gates where *k* is a small constant:




**Problem 6.** Find what CNOT looks like in the basis $\{|++\rangle, |+-\rangle, |-+\rangle, |--\rangle\}$. (write down the corresponding matrix representation.)

**Problem 7.** Exercise 4.26 from the Nielsen and Chuang book. Note that the first two rotations, from left, are $R_y(\pi/4)$ and the last two are $R_y(-\pi/4)$ where $R_y(\theta) = \exp(-i\sigma_Y\theta/2)$.

**Problem 8.** Using a bit-query black box, which acts as follows

$$|X\rangle \otimes |b\rangle \longrightarrow |X\rangle \otimes |b \oplus f(X)\rangle$$

make a phase-query black box with the following operation:

$$|X\rangle \longrightarrow (-1)^{f(X)}|X\rangle$$

where $X = b_1 b_2 \ldots b_n$ in the binary representation and $|X\rangle = |b_1\rangle|b_2\rangle\cdots|b_n\rangle$.