18.440: Lecture 5

Problems with all outcomes equally like, including a famous hat problem

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Outline

Equal likelihood

A few problems

Hat problem

A few more problems

Outline

Equal likelihood

- If a sample space S has n elements, and all of them are equally likely, the each one has to have probability $1/n$
- What is $P(A)$ for a general set $A \subset S$?
- Answer: $|A|/|S|$, where $|A|$ is the number of elements in A.

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Problems

- Roll two dice. What is the probability that their sum is three?
- Toss eight coins. What is the probability that exactly five of them are heads?
- In a class of 100 people with cell phone numbers, what is the probability that nobody has a number ending in 37?
- Roll ten dice. What is the probability that a 6 appears on exactly five of the dice?
- In a room of 23 people, what is the probability that two of them have a birthday in common?

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Recall the inclusion-exclusion identity

- $$P(\cup_{i=1}^{n} E_i) = \sum_{i=1}^{n} P(E_i) - \sum_{i_1<i_2} P(E_{i_1}E_{i_2}) + \ldots$$
$$+ (-1)^{(r+1)} \sum_{i_1<i_2<\ldots<i_r} P(E_{i_1}E_{i_2}\ldots E_{i_r})$$
$$= + \ldots + (-1)^{n+1} P(E_1 E_2 \ldots E_n).$$

- The notation $\sum_{i_1<i_2<i_r}$ means a sum over all of the $\binom{n}{r}$ subsets of size r of the set $\{1, 2, \ldots, n\}$.

Famous hat problem

- n people toss hats into a bin, randomly shuffle, return one hat to each person. Find probability nobody gets own hat.
- Inclusion-exclusion. Let E_i be the event that ith person gets own hat.
- What is $P(E_{i_1} E_{i_2} \ldots E_{i_r})$?
- Answer: $\frac{(n-r)!}{n!}$.
- There are $\binom{n}{r}$ terms like that in the inclusion exclusion sum. What is $\binom{n}{r}\frac{(n-r)!}{n!}$?
- Answer: $\frac{1}{r!}$.
- $P(\cup_{i=1}^n E_i) = 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \ldots \pm \frac{1}{n!}$
- $1 - P(\cup_{i=1}^n E_i) = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \ldots \pm \frac{1}{n!} \approx 1/e \approx .36788$

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Problems

- What's the probability of a full house in poker (i.e., in a five card hand, 2 have one value and three have another)?
- Answer 1:

$$\frac{\#\text{ ordered distinct-five-card sequences giving full house}}{\#\text{ ordered distinct-five-card sequences}}$$

- That's $\binom{5}{2} * 13 * 12 * (4 * 3 * 2) * (4 * 3)/(52 * 51 * 50 * 49 * 48) = 6/4165$.
- Answer 2:

$$\frac{\#\text{ unordered distinct-five-card sets giving full house}}{\#\text{ unordered distinct-five-card sets}}$$

- That's $13 * 12 * \binom{4}{3} * \binom{4}{2}/\binom{52}{5} = 6/4165$.
- What is the probability of a two-pair hand in poker?
- What is the probability of a bridge hand with 3 of one suit, 3 of one suit, 2 of one suit, 5 of another suit?

18.440 Probability and Random Variables

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