18.445 Introduction to Stochastic Processes

Lecture 21: Continuous time Markov chains

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Recall A point process N on $\mathbb{R}_+$ is called a Poisson process with intensity $\lambda > 0$ if

- For any $k \geq 1$, any $0 \leq t_1 \leq t_2 \leq \cdots \leq t_k$, the random variables $N(t_i, t_{i+1}], i = 1, ..., k-1$ are independent.
- For any interval $(a, b] \subset \mathbb{R}_+$, the variable $N(a, b]$ is a Poisson random variable with mean $\lambda(b-a)$.

Today's Goal :

- Characterization of Poisson process
- Continuous time Markov chain

Poisson process — Characterization

Theorem

Let $(X_t)_{t\geq 0}$ be an increasing right-continuous process taking values in $\{0, 1, 2, ...\}$ with $X_0 = 0$. Let $\lambda > 0$. Then the following statements are equivalent.

- $(X_t)_{t\geq 0}$ *is a Poisson process with intensity* λ.
- X *has independent increments, and as* $\epsilon \downarrow 0$, *uniformly in* t, *we have*

$$\mathbb{P}[X_{t+\epsilon} - X_t = 0] = 1 - \lambda\epsilon + o(\epsilon);$$

$$\mathbb{P}[X_{t+\epsilon} - X_t = 1] = \lambda\epsilon + o(\epsilon).$$

- X *has independent and stationary increments, and for all* $t \geq 0$ *we have* $X_t \sim$ *Poisson*(λt).

Continuous time Markov chains

Ω : countable state space

Definition

$(X_t)_{t\geq 0}$ is called a continuous time Markov chain if, for all $0 \leq t_1 \leq t_2 \leq \cdots \leq t_n \leq t_{n+1}$ and all $x_1, ...x_n, x_{n+1} \in \Omega$, we have

$$\mathbb{P}[X_{t_{n+1}} = x_{n+1} \,|\, X_{t_1} = x_1, ..., X_{t_n} = x_n] = \mathbb{P}[X_{t_{n+1}} = x_{n+1} \,|\, X_{t_n} = x_n]$$

moreover, the right hand side only depends on $(t_{n+1} - t_n)$.

Remark regularity requirement : the process is right-continuous, i.e. for all $t \geq 0$, there exists $\epsilon > 0$ such that $X_{t+s} = X_t$ for $s \in [0, \epsilon]$.

Semigroup of the chain

Definition

Suppose that $(X_t)_{t \geq 0}$ is a continuous time Markov chain. Define

$$P_t(x, y) = \mathbb{P}[X_t = y \mid X_0 = x].$$

(P_t) is called the semigroup of the chain.

- $P_0 = I$
- P_t is a stochastic matrix
- $P_{t+s} = P_t P_s$.

Examples

Example 1 Poisson process is Markovian.

$$P_s(x,y) = e^{\lambda s}\frac{(\lambda s)^{y-x}}{(y-x)!}.$$

Example 2 Let $(\hat{X}_n)_{n\geq 0}$ be a discrete time Markov chain with transition matrix Q. Let $(N_t)_{t\geq 0}$ be an independent Poisson process with intensity $\lambda > 0$. Define

$$X_t = \hat{X}_{N_t}, \quad t \geq 0.$$

Then $(X_t)_{t\geq 0}$ is a continuous time Markov chain.

$$P_s(x,y) = e^{-\lambda s}\sum_{k\geq 0}\frac{(\lambda s)^k}{k!}Q^k(x,y).$$

Holding times

Let $(X_t)_{t\geq 0}$ be a continuous time Markov chain.
Question : how long it stays at a state x ?
Define S_x to be the holding time at x :

$$X_0 = x, \quad S_x = \inf\{t \geq 0 : X_t \neq x\}.$$

Theorem

S_x has exponential distribution.

Lemma

Let T be a positive random variable. T has memoryless property :

$$\mathbb{P}[T > t + s \,|\, T > s] = \mathbb{P}[T > t]$$

if and only if T has exponential distribution.