# 18.445 Introduction to Stochastic Processes

## Lecture 22: Infinitesimal generator

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**Recall :** We consider continuous time Markov chain on countable state space with the following requirement

- (Homogeneity) $\mathbb{P}[X_{t+s} = y \,|\, X_s = x] = P_t(x, y)$
- (Right-continuity for the chain) For any $t \geq 0$, there exists $\epsilon > 0$, such that $X_{t+s} = X_t$ for all $s \in [0, \epsilon]$

**Today's Goal :**

- More words about the regularity of continuous time Markov chain
- Infinitesimal generator

# Jump process

Consider a continuous time Markov chain $(X_t)_{t\geq 0}$.
Define the **jump times** of the chain : $J_0, J_1, J_2, \ldots$

$$J_0 = 0, \quad J_{n+1} = \inf\{t > J_n : X_t \neq X_{J_n}\}, n \geq 0.$$

Define the **holding times** of the chain : $S_1, S_2, \ldots$

$$S_n = J_n - J_{n-1}, n \geq 1.$$

Define the jump process of the chain : $Y_0, Y_1, \ldots$

$$Y_n = X_{J_n}, n \geq 0.$$

- By right-continuity, we have $S_n > 0$.
- If $J_{n+1} = \infty$ for some $n$, set $X_\infty = X_{J_n}$

**Example** Let $(X_t)_{t\geq 0}$ be a Poisson process. Then
the jump process : $Y_n = n$
the holding times : $(S_n)_{n\geq 1}$ are i.i.d exponential.

# Explosion time

Define the **explosion time** $\xi$ by

$$\xi = \sup_n J_n = \sum_n S_n.$$

We only consider the chains with $\xi = \infty$.

**Summary** We consider continuous time Markov chain on countable state space with the following requirement

- (Homogeneity) $\mathbb{P}[X_{t+s} = y \mid X_s = x] = P_t(x, y)$
- (Right-continuity for the chain) For any $t \geq 0$, there exists $\epsilon > 0$, such that $X_{t+s} = X_t$ for all $s \in [0, \epsilon]$
- (Non explosion) The explosion time $\xi = \infty$

# Continuous time Markov chain

**Summary** We consider continuous time Markov chain on countable state space with the following requirement

- (Homogeneity) $\mathbb{P}[X_{t+s} = y \mid X_s = x] = P_t(x, y)$
- (Right-continuity for the chain) For any $t \geq 0$, there exists $\epsilon > 0$, such that $X_{t+s} = X_t$ for all $s \in [0, \epsilon]$
- (Non explosion) The explosion time $\xi = \infty$
- (Right-continuity in the semigroup) $P_\epsilon \to P_0 = I$ as $\epsilon \to 0$, pointwise for each entry.

Consider the transition semigroup $(P_t)_{t \geq 0}$

- $P_0 = I$
- $P_t$ is stochastic for all $t \geq 0$
- $P_{t+s} = P_t P_s$
- $P_\epsilon \to P_0 = I$ as $\epsilon \downarrow 0$

**Remark** Combining (3) and (4), the semigroup is right continuous for all $t$.

# Infinitesimal generator

### Theorem

*Let $(P_t)_{t\geq 0}$ be a right-continuous transition semigroup.*

- *For any state $x$, the limit exists*
$$q_x = \lim_{\epsilon\downarrow 0}(1 - P_\epsilon(x,x))/\epsilon \geq 0.$$
- *For any distinct states $x, y$, the limit exists*
$$q_{xy} = \lim_{\epsilon\downarrow 0} P_\epsilon(x,y)/\epsilon \geq 0.$$

### Lemma

*Let $f : (0,\infty) \to \mathbb{R}$ be a nonnegative function such that $\lim_{\epsilon\downarrow 0} f(\epsilon) = 0$, and assume that $f$ is subadditive, that is,*
$$f(t+s) \leq f(t) + f(s), \quad \forall t, s \geq 0.$$
*Then the limit $\lim_{\epsilon\downarrow} f(\epsilon)/\epsilon$ exists and equals $\sup_{t>0} f(t)/t$.*

# Infinitesimal generator

**Definition**

Set

$$q_{xx} = -q_x = \lim_{\epsilon \downarrow 0} (P_\epsilon(x,x) - 1)/\epsilon, \quad q_{xy} = \lim_{\epsilon \downarrow 0} P_\epsilon(x,y)/\epsilon.$$

Then the matrix $A = (q_{xy})_{x,y \in \Omega}$ is called the infinitesimal generator of the semigroup.

- $q_{xx} \leq 0$
- $q_{xy} \geq 0$ for $y \neq x$
- $\sum_y q_{xy} = 0$

# Examples

**Example 1** Let $(X_t)_{t\geq 0}$ be the Poisson process with intensity $\lambda > 0$. Then

$$q_{ii} = -\lambda, \quad q_{i,i+1} = \lambda.$$

**Example 2** Let $(\hat{X}_n)_{n\geq 0}$ be a discrete time Markov chain with transition matrix $Q$. Let $(N_t)_{t\geq 0}$ be an independent Poisson process with intensity $\lambda > 0$. Define

$$X_t = \hat{X}_{N_t}, \quad t \geq 0.$$

Then $(X_t)_{t\geq 0}$ is a continuous time Markov chain with generator $A = \lambda(Q - I)$.

# Infinitesimal generator and the jumping process

Recall : $(X_t)_{t\geq 0}$ is a continuous time Markov chain starting from $X_0 = x$.

$$J_1 = \inf\{t : X_t \neq x\}, \quad Y_1 = X_{J_1}.$$

**Theorem**

*For $x \neq y$, we have*

$$\mathbb{P}_x[J_1 > t, X_{J_1} = y] = e^{-q_x t}\frac{q_{xy}}{q_x}.$$

*In particular,*

- $\mathbb{P}_x[J_1 > t] = e^{-q_x t}$
- $\mathbb{P}_x[X_{J_1} = y] = q_{xy}/q_x$
- *$J_1$ and $X_{J_1}$ are independent.*

**Remark** : if $q_x = 0$, we say that $x$ is absorbing.