18.600: Lecture 24

Covariance and some conditional expectation exercises

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Outline

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Covariance and correlation

Paradoxes: getting ready to think about conditional expectation

A property of independence

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- Since $f(x,y) = f_X(x)f_Y(y)$ this factors as $\int_{-\infty}^{\infty} h(y)f_Y(y)dy \int_{-\infty}^{\infty} g(x)f_X(x)dx = E[h(Y)]E[g(X)]$.

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- Special case:

$$\mathrm{Var}(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} \mathrm{Var}(X_i) + 2 \sum_{(i,j):i<j} \mathrm{Cov}(X_i, X_j).$$

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- Are uncorrelated random variables always independent?
- No. Uncorrelated just means $E[(X - E[X])(Y - E[Y])] = 0$, i.e., the outcomes where $(X - E[X])(Y - E[Y])$ is positive (the upper right and lower left quadrants, if axes are drawn centered at $(E[X], E[Y])$) balance out the outcomes where this quantity is negative (upper left and lower right quadrants). This is a much weaker statement than independence.

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- Reduces problem to computing $\mathrm{Cov}(X_i, X_j)$ (for $i \neq j$) and $\mathrm{Var}(X_i)$.

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- Fairly dark as math humor goes (and no offense intended to anyone...) but dilemma is interesting.

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- You offer me a deal. I give you sack 1, you give me sacks 2 and 3. I give you sack 2 and you give me sacks 4 and 5. On the nth stage, I give you sack n and you give me sacks $2n$ and $2n+1$. Continue until I say stop.

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- In both stories, make infinitely many good trades and end up with less than I started with. "Paradox" is existence of 2-to-1 map from (smaller set) $\{2, 3, \ldots\}$ to (bigger set) $\{1, 2, \ldots\}$.

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- Every pile is bigger after transfer (and this can be true even if banker takes a portion of each pile as a fee).
- Banker seemed to make you richer (every pile got bigger) but really just reshuffled your infinite wealth.

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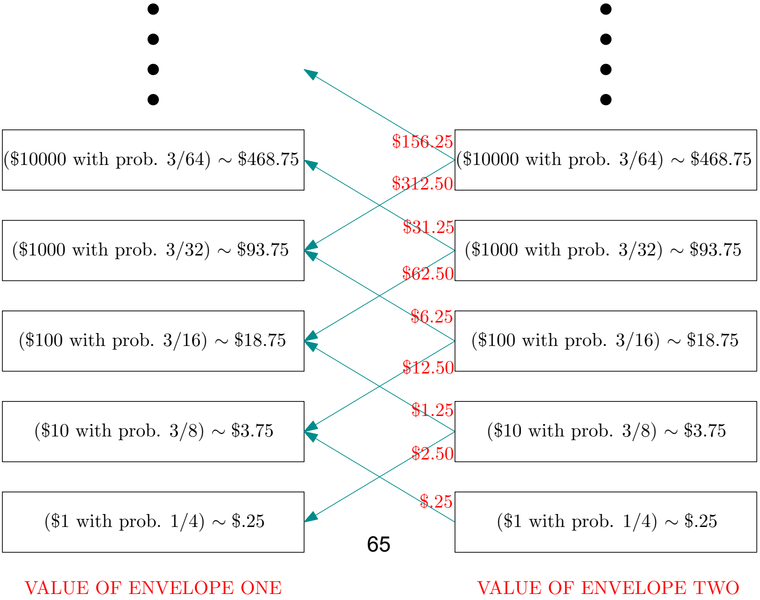
- Kind of a disguised version of money pile paradox. But more subtle. One has to replace "jth pile of money" with "restriction of expectation sum to scenario that first chosen envelop has 10^j". Switching indeed makes each pile bigger.
- However, "Higher expectation given amount in first envelope" may not be right notion of "better." If S is payout with switching, T is payout without switching, then S has same law as $T - 1$. In that sense S is worse.

Two envelope paradox



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- They can lead to strange conclusions, sometimes related to "reshuffling infinite (actual or expected) wealth to create more" paradoxes.
- Paradoxes can arise even when total transaction is finite with probability one (as in envelope problem).

18.600 Probability and Random Variables
Fall 2019