Fall 2012 18.440 Final Exam: 100 points
Carefully and clearly show your work on each problem (without writing anything that is technically not true) and put a box around each of your final computations.

1. (10 points) Lisa's truck has three states: broken (in Lisa's possession), working (in Lisa's possession), and in the shop. Denote these states B, W, and S.

(i) Each morning the truck starts out B, it has a 1/2 chance of staying B and a 1/2 chance of switching to S by the next morning.

(ii) Each morning the truck starts out W, it has 9/10 chance of staying W, and a 1/10 chance of switching to B by the next morning.

(iii) Each morning the truck starts out S, it has a 1/2 chance of staying S and a 1/2 chance of switching to W by the next morning.

Answer the following

(a) Write the three-by-three Markov transition matrix for this problem.

ANSWER: Ordering the states B, W, S, we may write the Markov chain matrix as

$$M = \begin{pmatrix} .5 & 0 & .5 \\ .1 & .9 & 0 \\ 0 & .5 & .5 \end{pmatrix}.$$

(b) If the truck starts out W on one morning, what is the probability that it will start out B two days later? **ANSWER:** $(9/10)(1/10) + (1/10)(1/2) = .09 + .05 = .14$

(c) Over the long term, what fraction of mornings does the truck start out in each of the three states, B, S, and W? **ANSWER:** We find the stationarity probability vector $\pi = (\pi_B, \pi_W, \pi_S) = (1/7, 5/7, 1/7)$ by solving $\pi M = \pi$ (with components of π summing to 1).

2. (10 points) Suppose that $X_1, X_2, X_3, \ldots$ is an infinite sequence of independent random variables which are each equal to 1 with probability 1/2 and -1 with probability 1/2. Write $Y_n = \sum_{i=1}^n X_i$. Answer the following:

(a) What is the probability that Y_n reaches 10 before the first time that it reaches -30? **ANSWER:** Probability p satisfies $10p + (-30)(1-p) = 0$, so $40p = 30$ and $p = 3/4$.

(b) In which of the cases below is the sequence Z_n a martingale? (Just circle the corresponding letters.)

(i) $Z_n = X_n + Y_n$ **ANSWER: NO**

(ii) $Z_n = \prod_{i=1}^n (2X_i + 1)$ **ANSWER: YES**

(iii) $Z_n = \prod_{i=1}^n (-X_i + 1)$ **ANSWER: YES**

(iv) $Z_n = \sum_{i=1}^n Y_i$ **ANSWER: NO**

(v) $Z_n = \sum_{i=2}^n X_i X_{i-1}$ **ANSWER: YES**

3. (10 points) Ten people throw their hats into a box and then randomly redistribute the hats among themselves (each person getting one hat, all 10! permutations equally likely). Let N be the number of people who get their own hats back. Compute the following:

(a) $E[N^2]$ **ANSWER:** Let N_i be 1 if ith person gets own hat, zero otherwise. Then
$E[(\sum N_i)^2] = \sum_{i=1}^{10} \sum_{j=1}^{10} E[N_i N_j] = 90(1/90) + 10(1/10) = 2.$

(b) $P(N = 8)$ **ANSWER:** There are $\binom{10}{2}$ ways to pick a pair of people to have swapped hats. So answer is $\binom{10}{2}/10!$.

4. (10 points) When Harry's cell phone is on, the times when he receives new text messages form a Poisson process with parameter $\lambda_T = 3$/minute. The times at which he receives new email messages form an independent Poisson process with parameter $\lambda_E = 1$/minute. He receives personal messages on Facebook as an independent Poisson process with rate $\lambda_F = 2$/minute.

(a) After catching up on existing messages one morning, Harry begins to wait for new messages to arrive. Let X be the amount of time (in minutes) that Harry has to wait to receive his first text message. Write down the probability density function for X. **ANSWER:** time is exponential with parameter $\lambda_T = 3$, so density function is $f(x) = 3e^{-3x}$ for $x \geq 0$.

(b) Compute the probability that Harry receives 10 new messages total (including email, text, and Facebook) during his first two minutes of waiting. **ANSWER:** Number total in two minutes is Poisson with rate $\lambda = 2(\lambda_E + \lambda_T + \lambda_F) = 12$. So answer is $\lambda^k e^{-\lambda}/k! = 12^{10}e^{-12}/10!$.

(c) Let Y be the amount of time elapsed before the third email message. Compute $\mathrm{Var}(Y)$. **ANSWER:** Variance of time till email message is $1/\lambda_E^2 = 1$. Memoryless property and additivity of variance of independent sums gives $\mathrm{Var}(S) = 3$.

(d) What is the probability that Harry receives no messages of any kind during his first five minutes of waiting? **ANSWER:** Time till first message is exponential with parameter 6. Probability this time exceeds 5 is e^{-30}.

5. (10 points) Suppose that X and Y have a joint density function f given by

$$f(x,y) = \begin{cases} 1/\pi & x^2+y^2<1 \\ 0 & x^2+y^2 \geq 1 \end{cases}.$$

(a) Compute the probability density function f_X for X. **ANSWER:**

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy = \begin{cases} \frac{1}{\pi}2\sqrt{1-x^2} & -1 \leq x \leq -1 \\ 0 & \text{otherwise} \end{cases}$$

(b) Express $E[\sin(XY)]$ as a double integral. (You don't have to explicitly evaluate the integral.) **ANSWER:**

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi}\sin(xy)dydx.$$

6. (10 points) Let X be the number on a standard die roll (i.e., each of $\{1,2,3,4,5,6\}$ is equally likely) and Y the number on an independent standard die roll. Write $Z = X + Y$.

(a) Compute the conditional probability $P[X = 6|Z = 8]$. **ANSWER:** 1/5

(b) Compute the conditional expectation $E[Y|Z]$ as a function of Z (for $Z \in \{2,3,4,\dots,12\}$). **ANSWER:**
$Z = E[Z|Z] = E[X+Y|Z] = E[X|Z] + E[Y|Z]$. By symmetry, $E[X|Z] = E[Y|Z] = Z/2$.

7. (10 points) Suppose that X_i are i.i.d. random variables, each of which assumes a value in $\{-1,0,1\}$, each with probability 1/3.

(a) Compute the moment generating function for X_1. **ANSWER:**
$Ee^{tX_1} = (e^{-t}+1+e^t)/3$.

(b) Compute the moment generating function for the sum $\sum_{i=1}^n X_i$.
ANSWER: $(e^{-t}+1+e^t)^n/3^n$

8. (10 points) Let X and Y be independent random variables. Suppose X takes values in $\{1,2\}$ each with probability $1/2$ and Y takes values in $\{1,2,3,4\}$ each with probability $1/4$. Write $Z = X+Y$.

(a) Compute the entropies $H(X)$ and $H(Y)$. **ANSWER:** $\log 2 = 1$ and $\log 4 = 2$.

(b) Compute $H(X,Z)$. **ANSWER:** $\log 2 + \log 4 = \log 8 = 3$.

(c) Compute $H(X+Y)$. **ANSWER:**

$$\sum_{i=1}^{6} P(X+Y=i)(-\log P(X+Y=i)) = 2\cdot\frac{1}{8}\log 8 + 3\cdot\frac{1}{4}\log 4 = 6/8+6/4 = 9/4.$$

9. (10 points) Let X be a normal random variable with mean 0 and variance 1.

(a) Compute $\mathbb{E}[e^X]$. **ANSWER:**

$$E(e^X) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} e^x dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(x^2-2x+1)/2+1/2} dx =$$

$$e^{1/2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(x-1)^2/2} dx = e^{1/2}.$$

(b) Compute $\mathbb{E}[e^X 1_{X>0}]$. **ANSWER:**

$$\begin{aligned} E(e^X 1_{X>0}) &= \int_{0}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} e^x dx \\ &= \int_{0}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(x^2-2x+1)/2+1/2} dx \\ &= e^{1/2} \int_{0}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(x-1)^2/2} dx \\ &= e^{1/2} \int_{-1}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = e^{1/2}(1-\Phi(-1)) = e^{1/2}\Phi(1), \end{aligned}$$

where $\Phi(a) = \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$.

(c) Compute $\mathbb{E}[X^2+2X-5]$. **ANSWER:**
$E[X^2] + 2E[X] - 5 = 1 + 0 - 5 = -4$.

10. (10 points) Let X be uniformly distributed random variable on $[0, 1]$.

(a) Compute the variance of X. **ANSWER:** $E[X^2] = \int_0^1 x^2 dx = 1/3$ and $E[X] = 1/2$ so $\text{Var}[X] = E[X^2] - E[X]^2 = 1/12$.

(b) Compute the variance of $3X + 5$. **ANSWER:** $9\text{Var}[X] = 3/4$.

(c) If $X_1, \ldots, X_n$ are independent copies of X, and $Z = \max\{X_1, X_2, \ldots, X_n\}$, then what is the cumulative distribution function F_Z? **ANSWER:**

$$F_Z(a) = P\{Z \leq a\} = \prod_{i=1}^n P\{X_i \leq a\} = F_{X_1}(a)^n = \begin{cases} 0 & a < 0 \\ a^n & a \in [0, 1] \\ 1 & a > 1 \end{cases}$$