MIT 18.642
Linear Algebra

Dr. Kempthorne

Fall 2024

Linear Algebra: Basic Concepts

Vectors

- Vector $\mathbf{v} \in R^m$
- Ordered list of m numbers: $\mathbf{v} = (v_1, v_2, \ldots, v_m)$
- Column vector / 1-column matrix:

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix}$$

- Special vectors:

$$\text{Zero vector } \mathbf{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \text{Ones vector } \mathbf{1} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

- Graphical Representation:

  Point $\mathbf{v} \in R^m$ or

  Directed segment: from $\mathbf{0}$ to $\mathbf{v}$

Basic Concepts: Vectors

Examples of Vectors:

- Closing prices on a given day t of all stocks in the S&P 500 stock index
$$\mathbf{p} = \begin{bmatrix} p_1(t) \\ \vdots \\ p_{500}(t) \end{bmatrix} \in R_{+}^{500}$$
where for stock $j(= 1, \ldots, 500)$ the series of daily closing prices is given by the time series $\{p_j(t)$, over days $t\}$
- For a portfolio holding only S&P 500 stocks, the number of shares held at start of day t
$$\mathbf{q} = \begin{bmatrix} q_1(t) \\ \vdots \\ q_{500}(t) \end{bmatrix}$$
- Value of the portfolio at end of day t
$$V = \sum_{j=1}^{500} q_j(t) P_j(t).$$

Vectors for Portfolios

- Include cash as asset $i = 0$ with value $p_0(t) \equiv \$1$ and time t cash position $q_0(t)$

$$\mathbf{p} = \begin{bmatrix} p_0(t) \\ p_1(t) \\ \vdots \\ p_{500}(t) \end{bmatrix} \in R_+^{501} \qquad \mathbf{q} = \begin{bmatrix} q_0(t) \\ q_1(t) \\ \vdots \\ q_{500}(t) \end{bmatrix}$$

- Portfolio Value at end of day t (no intra-day trading)

$$V_t = \sum_{j=0}^{500} q_j(t) p_j(t).$$

- Rebalance portfolio at end of day t, trade $\Delta_j(t)$ shares of each asset j with no net contribution/distribution:

$$q_j(t+1) = q_j(t) + \Delta_j(t) \text{ subject to } \sum_{j=0}^{500} \Delta_j(t) p_j(t) = 0.$$

  Note: $V_t = \sum_{j=0}^{500} q_j(t+1) p_j(t)$

- Portfolio Net Gain (PnL) on day $t+1$:

$$PnL(t+1) \quad = \quad V_{t+1} - V_t = \sum_1^n q_j(t+1)[p_j(t+1) - p_j(t)]$$

Vector Algebra

- Scalar multiplication: for m-vector $\mathbf{v} \in R^m$ and scalar $c \in R$,

$$c\mathbf{v} = c \times \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix} = \begin{bmatrix} c \times v_1 \\ c \times v_2 \\ \vdots \\ c \times v_m \end{bmatrix}$$

- Addition of vectors: for two m-vectors $\mathbf{v}$ and $\mathbf{w} \in R^m$,

$$\mathbf{v} + \mathbf{w} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{bmatrix} = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \\ \vdots \\ v_m + w_m \end{bmatrix}$$

- Inner Product / Dot Product: for m-vectors $\mathbf{v}$ and $\mathbf{w} \in R^m$,

$$\begin{aligned} \langle\langle \mathbf{v}, \mathbf{w} \rangle\rangle = \mathbf{v} \cdot \mathbf{w} &= v_1 w_1 + v_2 w_2 + \cdots v_m w_m \\ &= \sum_{j=1}^{m} v_j w_j \end{aligned}$$

Vector Algebra with Portfolios

- Two portfolios: $(m+1)$-vectors of cash and share holdings for each portfolio $(m = 501)$ start of day t (end of day $t-1$):

$$\mathbf{q}_t = \begin{bmatrix} q_0(t) \\ q_1(t) \\ \vdots \\ q_{500}(t) \end{bmatrix} \text{ and } \mathbf{w}_t = \begin{bmatrix} w_0(t) \\ w_1(t) \\ \vdots \\ w_{500}(t) \end{bmatrix}$$

- Beginning of Day (BOD) Values of two portfolios:
  $V\_BOD(\mathbf{q}_t, t) = \langle\langle \mathbf{q}_t, \mathbf{p}_{t-1} \rangle\rangle = \mathbf{q}_t \cdot \mathbf{p}_{t-1}$
  $V\_BOD(\mathbf{w}_t, t) = \langle\langle \mathbf{w}_t, \mathbf{p}_{t-1} \rangle\rangle = \mathbf{w}_t \cdot \mathbf{p}_{t-1}$
- End of Day (EOD) Values of two portfolios:
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  $V\_EOD(\mathbf{w}_t, t) = \langle\langle \mathbf{w}_t, \mathbf{p}_{t} \rangle\rangle = \mathbf{w}_t \cdot \mathbf{p}_{t}$
- Portfolio Net Gain (PnL) of two portfolios

$$\begin{aligned} PnL(\mathbf{q}_t, t) &= V\_EOD(\mathbf{q}_t, t) - V\_BOD(\mathbf{q}_t, t) \\ &= \mathbf{q}_t \cdot [\mathbf{p}_t - \mathbf{p}_{t-1}] \\ PnL(\mathbf{w}_t, t) &= \mathbf{w}_t \cdot [\mathbf{p}_t - \mathbf{p}_{t-1}] \end{aligned}$$

Vector Algebra with Portfolios

- Difference of two portfolios: Long/Short Portfolio

$$\mathbf{d}_t = \mathbf{q}_t - \mathbf{w}_t$$

$$\begin{aligned} PnL(\mathbf{d}_t, t) &= V\_EOD(\mathbf{d}_t, t) - V\_BOD(\mathbf{d}_t, t)]] \\ &= \mathbf{d}_t \cdot \mathbf{p}_t - \mathbf{d}_t \cdot \mathbf{p}_{t-1} \\ &= PnL(\mathbf{q}_t, t) - PnL(\mathbf{w}_t, t) \end{aligned}$$

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- Zero-Cost Portfolio $\mathbf{d}_t$:

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- Zero-Cost **Arbitrage** Portfolio $\mathbf{d}_t^*$:

$$\begin{aligned} V\_BOD(\mathbf{d}_t^*, t) &= 0. \\ PnL(\mathbf{d}_t^*, t) &> 0 \end{aligned}$$

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Issue(!): At time $(t-1)$

Prices $\mathbf{p}_t$ are Random.

$\Rightarrow PnL(\mathbf{d}_t^*, t)$ is Random.

Basic Concepts: Vectors

Vector Algebra

- Norm of a Vector: $\mathbf{v} \in R^m$.
$$\begin{aligned} ||\mathbf{v}|| &= \sqrt{\mathbf{v} \cdot \mathbf{v}} \\ &= \sqrt{\sum_{j=1}^{m} v_j^2} \end{aligned}$$
- Geometric definition of dot product / inner product
$$\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}| \times |\mathbf{w}| \times cos(\theta)$$
where θ is the angle between **v** and **w**
- Orthogonal vectors: **v** and **w** are **orthogonal** if
$$\mathbf{v} \cdot \mathbf{w} = 0.$$

Linear Independence

- Vectors $\mathbf{v}$ and $\mathbf{w}$ are **linearly independent** if
  $$c_1\mathbf{v} + c_2\mathbf{w} = \mathbf{0}$$
  only if $c_1 = c_2 = 0$

Vector Space: $\mathcal{S} = \{ \text{ vectors } \}$

- If $\mathbf{v} \in \mathcal{S}$ then $c\mathbf{v} \in \mathcal{S}$, for all scalars $c \in R$.
- If $\mathbf{v} \in \mathcal{S}$ and $\mathbf{w} \in \mathcal{S}$ then $(\mathbf{v} + \mathbf{w}) \in \mathcal{S}$

Basis for a Vector Space: $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is a basis for $\mathcal{S}$ if

- For any $\mathbf{w} \in \mathcal{S}$ there exist $c_1, c_2, \ldots c_p \in R$
  $$\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p$$
- If $\mathbf{w} = \mathbf{0}$ then $c_1 = c_2 = \cdots = c_p = 0$.

  Basis vectors are linearly independent

  $p = Dim(\mathcal{S})$

Basic Concepts: Matrices

Def: An m by n matrix A is a rectangular array of $(m \times n)$ numbers $\{a_{i,j}, 1 \le i \le m,\ \text{and } 1 \le j \le n\}$

$$
\begin{aligned}
A &= \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots\ddots\ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix} \\
&= ||a_{i,j}|| \ (m \times n)
\end{aligned}
$$

Equivalent representations:

- Array of n column vectors: $A = [\mathbf{a}_1\ \mathbf{a}_2\ \cdots\ \mathbf{a}_n]$ with

$$
\mathbf{a}_j = \begin{bmatrix} a_{1,j} \\ a_{2,j} \\ \cdots \\ a_{m,j} \end{bmatrix} \in R^m,\ j = 1, 2, \ldots, n
$$

Basic Concepts: Matrix Algebra

- Scalar multiplication: for a matrix $A = ||a_{ij}||$, scalar $c \in R$,
$$\begin{aligned} cA &= ||ca_{ij}|| = c\,[\mathbf{a}_1\ \mathbf{a}_2\ \cdots\ \mathbf{a}_n] \\ &= [c\mathbf{a}_1\ c\mathbf{a}_2\ \ldots\ c\mathbf{a}_n] \end{aligned}.$$
- Matrix Transpose: for an m by n matrix $A = ||a_{ij}||$ the transpose of A is the n by m matrix $t(A) = A^\top = ||a_{ji}||$
- The transpose of each column vector $\mathbf{a}_j$ is a row vector:
$$\mathbf{a}_j^\top = \begin{bmatrix} a_{1,j} \\ \vdots \\ a_{m,j} \end{bmatrix}^\top = [a_{1j}\ a_{2j}\ \cdots\ a_{mj}]$$
- The matrix transpose $A^\top$ is the array of row-vectors
$$A^\top = [\mathbf{a}_1\ \cdots \mathbf{a}_n]^\top = \begin{bmatrix} \mathbf{a}_1^\top \\ \vdots \\ \mathbf{a}_n^\top \end{bmatrix}$$

Basic Concepts: Matrix Algebra

Multiplication of a Matrix and a Vector

- $A = ||a_{ij}|| = [\mathbf{a}_1 \; \cdots \mathbf{a}_n]$ an m by n matrix
- $\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \in R^n$, an $n-$vector
- The product of A times $\mathbf{v}$ is the m-vector

$$\mathbf{y} = A\mathbf{v} = \mathbf{a}_1 v_1 + \mathbf{a}_2 v_2 + \cdots \mathbf{a}_n v_n.$$

Linear combination of A's columns

Vector of dot-products of A's rows

$$\mathbf{y} = \begin{bmatrix} \vdots \\ y_i \\ \vdots \end{bmatrix}, \text{ with } y_i = \sum_{j=1}^{n} a_{i,j} v_j = row_i(A) \cdot \mathbf{v}$$

Multiplication of Two Matrices

- $A = ||a_{ij}|| = [\mathbf{a}_1 \ \cdots \ \mathbf{a}_n]$ an m by n matrix
- $B = ||b_{ij}|| = [\mathbf{b}_1 \ \cdots \ \mathbf{b}_p]$ an n by p matrix
- The product of A times B is the m by p matrix
$$C = AB = A\,[\mathbf{b}_1 \ \mathbf{b}_2 \ \cdots \ \mathbf{b}_p] = [A\mathbf{b}_1 \ A\mathbf{b}_2 \ \cdots A\mathbf{b}_p]$$
- $C = ||c_{ij}||$ is m by p with
$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$
- A and B must be "conformal"
(column-count of A = row-count of B)

Basic Concepts: Matrix Algebra

Matrix Operations

- Addition of matrices $A = ||a_{ij}||$ and $B||b_{ij}||$, both m by n matrices

  $C = A + B = ||c_{ij}||$, where $c_{ij} = a_{ij} + b_{ij}$
- Transpose of matrix sum:

  $(A + B)^\top = A^\top + B^\top$
- Transpose of matrix product:

  $(AB)^\top = B^\top A^\top$ (order reverses!)
- Associative/Distributive laws:

  $A(BC) = (AB)C$

  $A(B + C) = AB + AC$
- Matrix multiplication typically not commutative:

  $AB \neq BA$, generally

Special Matrices

- A is a **Symmetric** Matrix if
$$A^T = A, \text{ i.e., } a_{ij} = a_{ji} \text{ for all } 1 \leq i, j \leq n = m$$
- Zero matrix
$$\mathbf{0} = \mathbf{0}_{m \times n}$$
- Identity matrix: I_n
$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vec{0} & \vec{0} & \ddots & \vec{0} \\ 0 & 0 & \cdots & 1 \end{bmatrix} = [\mathbf{e}_1\ \mathbf{e}_2\ \cdots \mathbf{e}_n]$$
- Matrix of ones: $\mathrm{J} = ||J_{ij}||$, with $J_{ij} = 1$ for all i, j.
- Diagonal matrix (n by n matrix of all zeros except for diagonal)
$$D = diag(d_1, d_2, \ldots, d_n)$$

Stochastic Matrices

- A: $a_{i,j} \geq 0$, $1 \leq i, j \leq m = n$
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Markov Chain Model

- m possible states: $s = 1, 2, \ldots, m$
- S_t : state at time $t = 0, 1, 2, \ldots$
- Stationary transition probabilities
  $$a_{i,j} = P(S_{t+1} = i \mid S_t = j) \qquad \text{(same for all } t\text{)}$$
- $\vec{\pi}(t)$: m-vector of probabilities
  $$\pi_j(t) = P(S_t = j),\ j = 1, \ldots, m$$
- $\vec{\pi}(t+1) = A\,\vec{\pi}(t)$

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Markov Chain Dynamics

- $\vec{\pi}(t+2) = A\,\vec{\pi}(t+1) = A\,[A\vec{\pi}(t)] = A^2\vec{\pi}(t)$
- Given $\vec{\pi}(0) = \vec{\pi}_0$: $\qquad \vec{\pi}(t) = A^t\vec{\pi}_0, \quad t = 1, 2, \ldots$
- Does: $\lim_{t\to\infty} \vec{\pi}(t) = \vec{\pi}_*$ exist?

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Positive matrices (prices/costs)

- A: $a_{i,j} > 0$, $1 \leq i \leq m$, $1 \leq j \leq n$

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Single-Period Market Model

- n Assets: $i = 1, \ldots, n$
- Times: $t = 0$ (start) $\quad t = T$ (end)

Matrices in Applications

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- Asset Price Processes:

$$\mathcal{A} = \{A_t^1, A_t^2, \ldots, A_t^n,\ t = 0, T\}$$

  - A_0^j is known price of asset j at time $t = 0$
  - A_T^j is random price of asset j at time $t = T$

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m **Possible Scenarios at time** $t = T$**:** $\Omega = \{\omega_1, \ldots, \omega_m\}$

- Positive Price Matrix at $t = T$ (Scenario-Dependent) :

  $A = ||a_{i,j}||,$

  where $a_{i,j} = A_T^j(\omega_i) = Price(asset\ j \mid scenario\ \omega_i)$

Single-Period Market Model

Portfolio of Assets

- Portfolio holdings in assets: $\vec{q} = (q_1, \ldots, q_n)$
- Portfolio Cost at time $t = 0$:
  $V_0 = \sum_{i=1}^{n} A_0^i q_i = \vec{A}_0 \cdot \vec{q}$
- Portfolio Pay-Off at time $t = T$ in scenario ω_i :
  $V_T(\omega_i) = \sum_{j=1}^{n} A_T^i(\omega_i) q_j$

m-vector of Pay-Offs for all scenarios

$$\vec{V}_T = \begin{bmatrix} V_T(\omega_1) \\ V_T(\omega_2) \\ \vdots \\ V_T(\omega_m) \end{bmatrix} = \begin{bmatrix} \vec{A}_T(\omega_1) \cdot \vec{q} \\ \vec{A}_T(\omega_2) \cdot \vec{q} \\ \vdots \\ \vec{A}_T(\omega_m) \cdot \vec{q} \end{bmatrix} = A\vec{q}$$

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Arbitrage Portfolio $\vec{q}$

- Zero/Negative Cost: $V_0 = \vec{A}_0 \cdot \vec{q} \leq 0$.
- Positive/Non-Zero Payoff:

$$\begin{aligned} V_T(\omega_i) \quad & \geq 0, \quad \textit{all } i = 1, \ldots, m \\ & > 0 \quad \text{at least one } \ i \end{aligned}$$

Single-Period Market Model

Important Theorems/Concepts

- Conditions ensuring **No Arbitrage**
  - Analyze space of $(m+1)$-tuples: cost/Payoffs
    $$\mathcal{P} = \{[V_0, \vec{V}_T] \in R^{m+1}, \text{ for all portfolios } \vec{q} \in R^n.\}$$

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- **Market Completeness** (existence of portfolios realizing any pay-off vector $\vec{v} \in R^m$ at time T)
- Existence and uniqueness of **Pricing Measure** on scenarios:
  $$Q^* : Q^*(\omega_i) = q_i^* > 0, \sum_i q_i^* = 1,$$
  which prices assets at $t = 0$ by
  $$A_0^j = \alpha E^{Q^*}[A_T^j] = \alpha \sum_{i=1}^m A_T^j(\omega_i) q_i^*$$
  for every asset $j = 1, \ldots, n$ and
  the scalar $\alpha > 0$ is a **Discount Factor**

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  the scalar $\alpha > 0$ is a **Discount Factor**
  - **No Arbitrage** if all $q_j^* > 0$
  - **Market Complete with No Arbitrage**
    if Pricing Measure Q^* is **Unique**.

Systems of Linear Equations

Solving a System of Equations:

$A\vec{x} = \vec{b}$ where

- A is $m \times n$ (known)
- $\vec{b} \in R^m$ (known)
- $\vec{x} \in R^n$ (unknown)

Case 1: $m = n$ and A has full rank n.

- $det(A) = |A| \neq 0$.
- A^{-1} exists:

$$AA^{-1} = A^{-1}A = \mathrm{I}_n.$$

- Solution:

$$\vec{x} = A^{-1}\vec{b}$$

Other Cases:

- $m < n$ (under-determined)
- $m > n$ (over-determined)
- $rank(A) < min(m, n)$

Eigenvalues and Eigenvectors

Definition 1: Let $A = ||a_{ij}||$, an n by n matrix of real values a_{ij}
Suppose

$$\begin{aligned} \lambda &\in R \\ \vec{v} &\in R^n \\ A\vec{v} &= \lambda\vec{v} \end{aligned}$$

Then:

- λ is an **eigenvalue** of A
- $\vec{v} \in R^n$ is an **eigenvector** of A corresponding to λ

Solving for Eigenvalues/Eigenvectors:

- $(A - \lambda\mathrm{I})\vec{v} = \vec{0}$.
  System of equations (linear in $\vec{v}$)
- $det(A - \lambda\mathrm{I}) = |A - \lambda\mathrm{I}| = 0$.
  Roots of polynomial in λ of degree n
  (May be real/complex/repeated)

Eigenvalues and Eigenvectors

Theorem. $max_{|\vec{v}|=1}|A\vec{v}| = max\{|\lambda(A)|,\}$
(maximum absolute eigenvalue of A)

Proof: (to come)

Theorem. Suppose that the n by n matrix A has n independent eigenvectors $\{\vec{v}_j, j = 1, \dots, n\}$

$$\begin{aligned} A\vec{v}_1 &= \lambda_1\vec{v}_1 \\ A\vec{v}_2 &= \lambda_2\vec{v}_2 \\ &\vdots \\ A\vec{v}_n &= \lambda_n\vec{v}_n \end{aligned}$$

Define the n by n matrix $S = [\vec{v}_1\ \vec{v}_2\ \cdots\ \vec{v}_n]$

- $AS = S\Lambda$, where $\Lambda = diag(\lambda_1, \dots, \lambda_n)$
- S^{-1}, the inverse matrix of S exists
- $A = S\Lambda S^{-1}$
- $S^{-1}AS = \Lambda$, (A is diagonalized by S)

Powers of a Diagonalizable Matrix A

- $A = S\Lambda S^{-1}$, where
  $S = [\vec{v}_1 \ \vec{v}_2 \ \cdots \ \vec{v}_n]$
  $\Lambda = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$
- $A^k = S\Lambda^k S^{-1}$, for positive integer k
  Eigenvectors: columns of S (unchanged!)
  Eigenvalues: $\lambda_j(A^k) = [\lambda_j(A)]^k = \lambda_j^k$

Eigenvalues and Eigenvectors

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State Equations in Kalman Filters

- $\vec{u}_t$: the n-dimensional state of a dynamical system at time t
- $\vec{u}_0$, the system state at time $t = 0$.
- State equation (deterministic):

$$\vec{u}_t = A\vec{u}_{t-1}, \ t = 1, 2, \ldots$$

General Solution: $\vec{u}_t = A^t \vec{u}_0$.

State Equations in Kalman Filters

State Equations:

- $\vec{u}_0$: initial state
- $\vec{u}_t = A\vec{u}_{t-1},\ t = 1, 2, \ldots$
- $\vec{u}_t = A^t \vec{u}_0 = S\Lambda^t S^{-1} \vec{u}_0.$

where

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Explicit Solution:

- $\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n\}$ are independent (S^{-1} exists)
  $\Longrightarrow$ form a basis for R^n
- $\exists$ (unique) $c_1, c_2, \ldots, c_n$:
  $$\vec{u}_0 = c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_n \vec{v}_n$$

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$$\vec{u}_0 = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$$
$$\begin{aligned}\Longrightarrow \vec{u}_t = A^t\vec{u}_0 &= S\Lambda^t S^{-1}\vec{u}_0 \\ &= c_1\lambda_1^t\vec{v}_1 + c_2\lambda_2^t\vec{v}_2 + \cdots + c_n\lambda_n^t\vec{v}_n\end{aligned}$$

Key Theorem

Theorem. Let A be a Real Symmetric Matrix. Then the following holds:

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  $\vec{v}_1$ and $\vec{v}_2$ are orthogonal: $\vec{v}_1 \cdot \vec{v}_2 = 0$.
- A is orthonormally diagonalizable

  $A = S\Lambda S^{-1}$ where
  $\Lambda = diag(\lambda_1, \dots, \lambda_n)$
  $S = [\vec{v}_1 \ \vec{v}_2 \ \cdots \ \vec{v}_n]$
  $S^T = S^{-1}$, i.e., $S^T S = \mathrm{I}$

$$\vec{v}_i^{\,T} \vec{v}_j = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

Singular Value Decomposition

Theorem. Every m by n matrix A_{mn} of can be expressed as

$A_{mn} = U_{mm}\Sigma_{mn}V_{nn}^T$, where

- U_{mm} is an $m \times m$ orthogonal matrix ($U_{mm}^T = U_{mm}^{-1}$)
- Σ_{mn} is a (non-negative) diagonal matrix

$$\Sigma_{i,j} = \begin{cases} \sigma_j, & i = j \;\; (each\ \sigma_j \geq 0) \\ 0, & i \neq j \end{cases}$$

- V_{nn} is an $n \times n$ orthogonal matrix ($V_{nn}^T = V_{nn}^{-1}$)

Proof:

- Let $r \leq n \leq m$ be the rank of A.
- The symmetric matrix $A^T A$ has rank r.
- Let $\sigma_1^2, \sigma_2^2, \ldots, \sigma_r^2$ be the non-zero eigenvalues of $A^T A$.
- Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r$ be the orthonormal eigenvectors corresponding to the positive eigen values and let $\vec{v}_{r+1}, \ldots \vec{v}_n$ be orthonormal eigenvectors corresponding to the zero eigenvalues which are also orthogonal to the first r $\vec{v}_j$.

Singular Value Decomposition

Proof (continued)

- For $1 \leq j \leq r$, define $\quad \vec{u}_j = \frac{1}{\sigma_j} A\vec{v}_j$
- For any two i, j we have

$$
\begin{array}{rcl}
\vec{u}_i^T \vec{u}_j & = & \frac{1}{\sigma_i \sigma_j} \vec{v}_i^T A^T A \vec{v}_j \\
& = & \frac{1}{\sigma_i \sigma_j} \vec{v}_i^T (A^T A \vec{v}_j) \\
& = & \frac{1}{\sigma_i \sigma_j} \vec{v}_i^T (\sigma_j^2 \vec{v}_j) \\
& = & \frac{\sigma_j}{\sigma_i} \vec{v}_i^T \vec{v}_j = \delta_{i,j} \quad (1 \text{ if } i = j, \text{ else } 0\ )
\end{array}
$$

- Complete this collection of m-vectors to an orthonormal basis with $\vec{u}_{r+1}, \ldots, \vec{u}_m$

$$U = [\vec{u}_1 \ \vec{u}_2 \ \cdots \vec{u}_m]$$

- Define the $n \times n$ matrix

$$V = [\vec{v}_1 \ \vec{v}_2 \ \cdots \vec{v}_n]$$

Singular Value Decomposition

Proof (continued)

We can now write:

$$
\begin{aligned}
U^T AV &= U^T[A\vec{v}_1 \quad A\vec{v}_2 \quad \cdots \; A\vec{v}_n] \\
&= U^T[\sigma_1\vec{u}_1 \; \sigma_2\vec{u}_2 \; \cdots \; \sigma_n\vec{u}_n] \\
&= [\sigma_1 U^T\vec{u}_1 \mid \sigma_2 U^T\vec{u}_2 \mid \cdots \mid \sigma_n U^T\vec{u}_n] \\
&= [\sigma_1\vec{e}_1 \mid \sigma_2\vec{e}_2 \mid \cdots \mid \sigma_n\vec{e}_n] \\
&= \Sigma \quad \text{note: upper-left block } = diag(\sigma_1, \sigma_2, \ldots, \sigma_n)
\end{aligned}
$$

Which gives:

$$
\begin{aligned}
A &= (\mathrm{I}_m)A(\mathrm{I}_n) \\
&= (UU^T)A(VV^T) \\
&= (UU^T)A(VV^T) \\
&= U(U^T AV)V^T \\
&= U\Sigma V^T
\end{aligned}
$$

Singular Value Decomposition

Definition: The **singular values** of A are the square roots of the eigenvalues of $A^T A$. It is customary to order them:

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$$

(last $n - r$ are equal to zero)

$$\begin{aligned} A &= U\Sigma V^T \\ &= [\vec{u}_1 \mid \vec{u}_2 \mid \cdots \mid \vec{u}_n] diag(\sigma_1, \sigma_2, \ldots, \sigma_n)[\vec{v}_1 \mid \vec{v}_2 \mid \cdots \mid \vec{v}_n]^T \\ &= \sigma_1 \vec{u}_1 \vec{v}_1^T + \sigma_2 \vec{u}_2 \vec{v}_2^T + \cdots + \sigma_n \vec{u}_n \vec{v}_n^T \\ &= \sigma_1 \vec{u}_1 \vec{v}_1^T + \sigma_2 \vec{u}_2 \vec{v}_2^T + \cdots + \sigma_r \vec{u}_r \vec{v}_r^T \end{aligned}$$

(Sum of r rank-1 matrices)

Note: The **reduced SVD** (for $n \leq m$) can be written replacing the $m \times m$ U with the $m \times n$ sub-matrix consisting of the first n columns of U and replacing the $m \times n$ diagonal matrix Σ with the $n \times n$ upper-left sub matrix.

Perron-Frobenius Theorem

Theorem. Suppose $A = ||a_{i,j}||$ is an n by n Real Positive Matrix $(a_{i,j} > 0, \forall i, j)$. Then

- There is a real eigenvalue λ_0 such that all other eigenvalues satisfy
  $$|\lambda| < \lambda_0.$$
- There is a positive eigenvector $\vec{v}$ corresponding to λ_0
- λ_0 is an eigenvalue of multiplicity 1.

Proof:

- Consider $\mathcal{T} = \{t : A\vec{x} \geq t\vec{x},\ \text{for some } \vec{x} \geq 0\}$.
- Claim: $t_{max} = sup\{t \in \mathcal{T}\}$ is the real eigenvalue λ_0

Perron-Frobenius Theorem

Proof (continued)

- Fix any $\vec{x} \geq 0$ ($\vec{x} \neq \vec{0}$). Then
  $$A\vec{x} \geq t\vec{x} \text{ where } t = \min_j \{a_{j,j}\} > 0.$$
  For each component i of $A\vec{x}$
  $$\begin{aligned} [A\vec{x}]_i = \textstyle\sum_{j=1}^n a_{i,j}x_j &\geq a_{i,i}x_i \\ &\geq \min_j(\{a_{j,j}\})x_i = tx_i \end{aligned}$$
- The set $\mathcal{T} = \{t : A\vec{x} \geq t\vec{x}, \text{ for some } \vec{x} \geq 0\}$ must satisfy:
  $$\{t : 0 < t \leq \min_j\{a_{j,j}\}\} \subset \mathcal{T}$$
- So $t_{max} = sup\{t \in \mathcal{T}\} > 0$
- Claim: $\exists \vec{x} > 0 : A\vec{x} = t_{max}\vec{x}$.
  $$\Longrightarrow \lambda_0 = t_{max} \text{ is an eigenvalue with eigenvector } \vec{x}.$$

Proof (continued)

- Suppose that $\vec{x}$ is such that
$$A\vec{x} \geq t_{max}\vec{x} \text{ but } A\vec{x} \neq t_{max}\vec{x}$$
- Define $\vec{y} = A\vec{x} - t_{max}\vec{x} \geq \vec{0}$ (but $\neq \vec{0}$).
- $A\vec{y} > \vec{0}$ because A is strictly positive and $\vec{y} \geq \vec{0}$ and $\neq 0$. Consequently
$$\begin{aligned} A\vec{y} &> \vec{0} \\ \implies A(A\vec{x} - t_{max}\vec{x}) &> \vec{0} \\ \implies A(A\vec{x}) &> t_{max}(A\vec{x}) \end{aligned}$$

but this contradicts the definition of t_{max} because it can be increased using the n-vector $(A\vec{x})$ instead of $\vec{x}$,
It follows that there exists an $\vec{x} > \vec{0}$ (strictly) such that
$$A\vec{x} = t_{max}\vec{x}.$$

Proof (continued)
Still to show: no eigenvalue is larger than $\lambda_0 = t_{max}$
Suppose λ and $\vec{z}$ are an eigenvalue/eigenvector pair:

$$A\vec{z} = \lambda\vec{z}.$$

Note: λ may be complex and z may be complex and/or negative.

$$\lambda z = A\vec{z} \Longrightarrow |\lambda| \begin{bmatrix} |z_1| \\ |z_2| \\ \vdots \\ |z_n| \end{bmatrix} = \begin{bmatrix} |(Az)_1| \\ |(Az)_2| \\ \vdots \\ |(Az)_n| \end{bmatrix} \leq A \begin{bmatrix} |z_1| \\ |z_2| \\ \vdots \\ |z_n| \end{bmatrix}$$

The last inequality follows because $|\sum_j a_{i,j} z_j| \leq \sum_j a_{i,j}|z_j|$, for each i. Thus it follows that $t = |\lambda| \in \mathcal{T}$ with

$$\vec{x} = \begin{bmatrix} |z_1| \\ |z_2| \\ \vdots \\ |z_n| \end{bmatrix} \text{ such that } A\vec{x} \geq t\vec{x}. \text{ Thus } |\lambda| \leq t_{max} = \lambda_0.$$

Proof (continued)

Suppose that eigenvalue $\lambda_0 = t_{max}$ has multiplicity at least 2.

- Let $\vec{x}$ and $\vec{y}$ be two distinct eigenvectors of norm 1.
- Since $\vec{x}$ and $\vec{y}$ are both eigenvectors of eigenvalue λ_0, $\vec{x} - \vec{y}$ must also be an eigenvector of eigenvalue λ_0.
- Since $\vec{x}$ and $\vec{y}$ are distinct, the vector $\vec{x} - \vec{y}$ has both positive and negative entries.
- But this is impossible because we showed that the eigenvector of λ_0 must have entries all with the same sign.

Financial News Sources

- Yahoo Finance: `https://finance.yahoo.com/`
- Wall Street Journal: `https://www.wsj.com/`
- Bloomberg: `https://www.bloomberg.com/`
- Marketwatch: `https://www.marketwatch.com/`
- Reuters: `https://www.reuters.com/`
- Zerohedge: `https://www.zerohedge.com/`
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