18.642 Lecture Notes
One-Period Financial Models

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Contents

1 One-Period Economy with Two Assets

1.1 Prices and Returns

Consider a one-period economy beginning at $t = 0$ and ending at $t = T$. Suppose there are two assets:

- B: bond with prices B_t, at times $t = 0, T$

  At $t = 0$, the price B_0 is known. At the end of the period ($t = T$), the price of the bond is also known:

  $$B_T = B_0 \times (1 + r_f T).$$

  The bond is a risk-free asset which has an **absolute return**

  $$R_B = B_T - B_0 = B_0(r_f T)$$

  and a **percentage return**

  $$r_B = \frac{R_B}{B_0} = r_f T.$$

  Also, the **simple rate of return** is $r_f = \frac{1}{T}\frac{B_T - B_0}{B_0} = \frac{1}{T}(\frac{B_T}{B_0} - 1)$.

- Stock S: with prices S_t at times $t = 0, T$. At $t = 0$, the stock price S_0 is known. At the end of the period ($t = T$), the price of the stock is uncertain, depending on the state of the economy.

To model the state of the economy, let ω_t denote the state of the economy at time t. At the beginning of the period, the state of the economy ω_0 is known, i.e., the prices of B and S are known. At the end of the period ($t = T$), suppose the economy could have two states $\omega \in \Omega_T = \{d, u\}$:

$\omega = u$: market improves (stock goes *up*)
$\omega = d$: market declines (stock goes *down*)

and the price of the stock is

$$S_T = \begin{cases} S_T^u, & if \quad \omega_T = u \\ S_T^d, & if \quad \omega_T = d. \end{cases}.$$

where $S_T^d < S_T^u$.

The stock has an **absolute return** of $R_S = S_T - S_0$ and a **percentage return** of $r_S = \frac{S_T}{S_0} - 1$. These are uncertain, depending on the state of the economy at $t = T$.

1.2 Portfolios of Assets

Suppose an investor has a portfolio at the beginning of the period consisting of π_S shares of the stock S and π_B units of the bond B.

It is convenient to represent the portfolio as a 2-vector:

$$\vec{\pi} = \begin{bmatrix} \pi_B \\ \pi_S \end{bmatrix}.$$

Let V_0 be the value (cost) of the portfolio at $t = 0$ and V_T be the value (payoff) of the portfolio at $t = T$. These are given by the known value at $t = 0$:

$$V_0 = \pi_B B_0 + \pi_S S_0$$

and the state-dependent value at $t = T$:

$$V_T = \begin{cases} \pi_B B_T + \pi_S S_T^u, & if \quad \omega_T = u \\ \pi_B B_T + \pi_S S_T^d, & if \quad \omega_T = d \end{cases}$$

1.3 Contingent Claims and Replicating Portfolios

For the One-Period Simple economy, consider a **contingent claim** C with value/cost C_0 at $t = 0$ and payoff C_T at $t = T$. The payoff at $t = T$ is state dependent given by

$$\vec{C}_T = \begin{bmatrix} C_T^d \\ C_T^u \end{bmatrix}.$$

where we assume that C_T^d and C_T^u are known.

With respect to the contingent claim C, a portfolio

$$\vec{\pi} = \begin{bmatrix} \pi_B \\ \pi_S \end{bmatrix}$$

of the assets B and S is a **replicating portfolio** if it satisfies:

$$\begin{aligned} \pi_B B_T + \pi_S S_T^d &= C_T^d \\ \pi_B B_T + \pi_S S_T^u &= C_T^u \end{aligned}$$

Replicating portfolios of contingent claims are critical features of financial market models. Consider the following questions:

- For known,given payoffs of the contingent claim $\vec{C}_T = (C_T^u, C_T^d)$, depending on the state, solve this system of equations for π_S and π_B the units of B and S in the replicating portfolio.

- Under what conditions is (i) the solution unique; (ii) the solution is not unique; (iii) the solution does not exist. (These are linear algebra questions)

- What additional assumptions are required for *any* contingent claim to have a replicating portfolio?

  Allowing π_S and π_B to take arbitrary real numbers relates to unlimited liquidity (i.e., asset prices do not depend on quantity traded), divisibility of investments (i.e., allowing shares/unit π_B and π_S to be fractional), ability to short sell assets (i.e., allowing π_B and/or π_S to be negative).

Exercises (1-3):

1. In the general case of the One-Period Economy, prove that the hypothesis of no arbitrage is satisfied only if the following strict inequality is satisfied:

$$\frac{S_T^d}{1 + rT} < S_0 < \frac{S_T^u}{1 + r_f T}.$$

Hint: Consider a violation of either inequality and construct a portfolio and trading strategy with arbitrage (i.e., its cost C_0 at $t=0$ is lower than its payoffs at $t=T$).

2. Consider the special case of the simple economy with

$$B_0 = 100 \text{ and } B_T = 103.$$
$$S_0 = 100 \text{ and } S_T^u = 130,\ S_T^d = 90.$$

and the space of vector payoffs at $t=T$ of portfolios $\vec{\pi} = \begin{bmatrix} \pi_B \\ \pi_S \end{bmatrix}$

$$\vec{C}_T = \begin{bmatrix} \pi_B B_0 + \pi_S S_T^d \\ \pi_B B_0 + \pi_S S_T^u \end{bmatrix}, \quad \text{subject to constraints on } \vec{\pi}.$$

Consider the contingent claim P corresponding to the simple **Put** option with **Strike price** K=100. This option gives the holder the option to sell the stock S for strike price at $t=T$. The payoff at $t=T$ depends on the state and is given by

$$P_T = max(0, K - S_T)$$

(a) Find the portfolio $\vec{\pi}^* = (\pi_B^*, \pi_S^*)^T$ which replicates the payoffs at $t=T$ of the put option.

(b) Solve the $t=0$ price of the replicating portfolio in (a).

(c) Is the price in (b) an arbitrage-free price?
Hint: see problem 3.

3. In the general case of exercise 1, suppose the contingent claim (derivative) C is arbitrage free.

(a) Show that the cost/value of C at $t=0$ is given by
$$C_0 = \pi_B^* B_0 + \pi_S^* S_0$$

(b) This formula can be reexpressed as
$$C_0 = \left(\frac{S_0 - (1+rT)^{-1} S_T^d}{S_T^u - S_T^d}\right) C_T^u \; + \; \left(\frac{(1+rT)^{-1} S_T^u - S_0}{S_T^u - S_T^d}\right) C_T^d$$

(c) Note that
$$(1+rT)^{-1} S_T^u$$
is the discounted price of S_T at $t=T$ if $\omega = u$, and
$$(1+rT)^{-1} S_T^d$$
is the discounted price of S_T at $t=T$ if $\omega = d$.
Also, $S_0(1+rT)$ is the **Forward Value** of a bond B position bought at $t=0$ by selling one unit of stock S at $t=0$.
Show that the arbitrage-free cost of the contingent claim can be expressed as:
$$\begin{aligned} C_0 &= (1+rT)^{-1}\left[\left(\frac{(1+rT)S_0 - S_T^d}{S_T^u - S_T^d}\right) C_T^u \; + \; \left(\frac{S_T^u - (1+rT)S_0}{S_T^u - S_T^d}\right) C_T^d\right] \\ &= (1+rT)^{-1}[q^u C_T^u + q^d C_T^d] \end{aligned}$$

where

$$
\begin{aligned}
q^u &= \left(\frac{(1+rT)S_0 - S_T^d}{S_T^u - S_T^d}\right) \\
q^d &= \left(\frac{S_T^u - (1+rT)S_0}{S_T^u - S_T^d}\right).
\end{aligned}
$$

Note: the measure Q on $\Omega = \{d, u\}$ such that

$Q(u) = q^u = Prob(\omega = u)$, and

$Q(d) = q^d = Prob(\omega = d)$,

is a probability measure/distribution. It is called the **Pricing Measure** and the probabilities q^u, q^d (which sum to 1) are called **risk-neutral probabilities**. In the next secton we will see that contingent claims can be expressed as discounted expected payoffs with respect to the risk-neutral/Pricing measure

$C_0 = B_0 E^Q(C_T/B_T)$,

where the expection (E^P) for general probability measures P uses Q.

2 One-Period Economy: General Case

The simple one-period economy with two assets and two states extends to a general case of n assets and m states at period end.

2.1 Single-Period Financial Model

Consider the following setup and notation for the model of the economy:

One Period: The single period has duration T, with

period start ($t = 0$) and period end ($t = T$).

n **Assets:** $j = 1, 2, \ldots, n$

Asset Prices at Period Start ($t = 0$): $\{P_0^j, j = 1, \ldots, n\}$

These prices are known at $t = 0$. Define the $(1 \times n)$ matrix whose first row is the n-vector of prices:

Initial Price Matrix: $A_0 = \begin{bmatrix} P_0^1 \; P_0^2 \; \cdots \; P_0^n \end{bmatrix}$ $(1 \times n)$

State-Space at Period End ($t = T$): $\Omega_T = \{\omega_i = 1, 2, \ldots, m\}$

Let ω denote the state of the economy at time $t = T$. At time $t = 0$, the end-period state ω is uncertain. However, suppose that Ω_T, the space of possible possible states (or scenarios) is known and of size m.

Ultimately, the model of the simple economy could specify a probability measure on Ω_T, giving the real-world conditional probability distribution of ω as a random variable given $t = 0$. For now, we analyze the simple economy with knowledge of Ω_T, but no knowledge of the relative likelihood of different states at time $t = T$.

Prices at Period End: $\{P_T^j, j = 1, \ldots, n\}$

For each asset j, let P_T^j denote the asset price at time $t = T$. At time $t = 0$, the time $t = T$ asset price value P_T^j is a random variable. Suppose this random variable depends only on the state ω at time $t = T$. For now we focus on the possible outcomes of the random variables, not their relative likelihoods (which would be given by an assumed probability model on Ω_T).

Denote the price of asset j at time $t = T$ for state ω_i by:

$$A_{i,j} = P_T^j(\omega_i).$$

Define the time (t=T) price matrix A with dimension $(m \times n)$ as follows:

$$A = [\vec{a}_1 \; \vec{a}_2 \; \cdots \; \vec{a}_n] = \begin{bmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,n} \\ A_{2,1} & A_{2,2} & \cdots & A_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m,1} & A_{m,2} & \cdots & A_{m,n} \end{bmatrix}.$$

As indicated $\vec{a}_j$, the jth column-vector of A equals the m-vector of prices for asset j over the m states

$$\vec{a}_j = \begin{bmatrix} A_{1,j} \\ A_{2,j} \\ \vdots \\ A_{m,j} \end{bmatrix} = \begin{bmatrix} P_T^j(\omega_1) \\ P_T^j(\omega_2) \\ \vdots \\ P_T^j(\omega_m) \end{bmatrix} \in R^m.$$

For each asset j, the ith coordinate of $\vec{a}_j$ equals its price at $t = T$, if the $(t = T)$ state is ω_i.

Financial Model and Asset Price Processes:

Define the **Asset Price Process**:

$$\mathcal{A} = \{P_t^1, \ldots, P_t^n; t = 0, T\},$$

the collection of asset price processes for the n assets. The value P_0^j is the current price of the jth asset and P_T^j, the price of the jth asset at time T is a random variable determined by the state outcome $\omega \in \Omega_T$.

The **Single-Period Financial Model** $\mathcal{M} = (\Omega_T, \mathcal{A})$ is specified by the state-space Ω_T and the state-dependent prices of the n assets at time T.

2.2 Portfolios

The theory of asset pricing in single-period financial models works extensively with the pricing of portfolios. The following discussion develops useful terminology and notation.

Portfolio of Assets: Π and $\vec{\pi}$

Consider a portfolio of assets:

$$\text{Portfolio } \Pi = \{\pi_j \text{units of asset } j, j = 1, \ldots, n\}$$

where π_j denotes the unit quantity of asset j. If j is a stock then the units are shares, and if j is a bond, or financial contract (e.g., option, future, or derivative), the units are number of bonds or contracts.

Define $\vec{\pi}$ to be the n-vector of asset quantities:

$$\vec{\pi} = \begin{bmatrix} \pi_1 \\ \pi_2 \\ \vdots \\ \pi_n \end{bmatrix} \in R^n$$

When the meaning is clear, it will be convenient to equate the portfolio Π and the m-vector $\vec{\pi}$. Trades buying a portfolio take place only at time $t = 0$ and trades selling a portfolio take place only at time $t = T$.

Portfolio Cost (t=0) and Payoff (t=T): Values of Portfolio $\vec{\pi}$

To analyze a fixed portfolio $\vec{\pi}$ we work with its value $V_t(\vec{\pi})$, at period start $(t = 0)$ and end $(t = T)$. Under the assumption of zero transactions costs,

$V_0(\vec{\pi})$ is the **cost** of buying portfolio $\vec{\pi}$ at $t = 0$ and $V_T(\vec{\pi})$ is the **payoff** from selling portfolio $\vec{\pi}$ at $t = T$.

In the simple economy, the value of portfolio $\vec{\pi}$ at time $t = 0$ is known:

$$V_0(\vec{\pi}) = \sum_{j=1}^{n} \pi_j P_0^j = A_0 \vec{\pi}$$

This is the dot product of $\vec{\pi}$ and $\vec{A}_0$, the n-vector corresponding to the row of the $(1 \times n)$ matrix A_0.

The value of portfolio $\vec{\pi}$ at time $t = T$ is uncertain with state-dependent values given by the m-vector:

$$\begin{aligned}
\vec{V}_T(\vec{\pi}) &= \begin{bmatrix} V_T(\vec{\pi};\omega_1) \\ V_T(\vec{\pi};\omega_2) \\ \vdots \\ V_T(\vec{\pi};\omega_m) \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^{n} \pi_j P_T^j(\omega_1) \\ \sum_{j=1}^{n} \pi_j P_T^j(\omega_2) \\ \vdots \\ \sum_{j=1}^{n} \pi_j P_T^j(\omega_m) \end{bmatrix} \\
&= \sum_{j=1}^{n} \pi_j \vec{a}_j \\
&= A\vec{\pi}
\end{aligned}$$

Let us introduce the following notation for the row vectors of A:

$$\vec{A}_i = \begin{bmatrix} A_{i,1} & A_{i,2} & \cdots & A_{i,m} \end{bmatrix},\ i = 1, \ldots, n$$

We can write:

$$A = \begin{bmatrix} \vec{A}_1 \\ \vec{A}_2 \\ \vdots \\ \vec{A}_m \end{bmatrix}$$

If there are no transaction costs, the payoff from selling portfolio $\vec{\pi}$ at time $t = T$ in state ω_i is

$$V_T^i = \mathit{Payoff}_T(\vec{\pi}, \omega_i) = \vec{A}_i \cdot \vec{\pi}.$$

The m-vector of payoffs of portfolio $\vec{\pi}$ for each of the m states at time $t = T$ is

$$\vec{V}_T = \begin{bmatrix} V_T^1 \\ V_T^2 \\ \vdots \\ V_T^m \end{bmatrix} = \begin{bmatrix} \vec{A}_1\vec{\pi} \\ \vec{A}_2\vec{\pi} \\ \vdots \\ \vec{A}_m\vec{\pi} \end{bmatrix}.$$

This payoff/value vector is the vector of dot products of the row-vectors of A and the portfolio vector $\vec{\pi}$ (all n-vectors).

2.3 Contingent Claims

In context of the Single-Period Financial Model, define a **contingent claim** C to be financial security with price process $\{C_t, t = 0, T\}$.

$$\begin{aligned}
C_0 &= \text{Price of } C \text{ at } t = 0. \\
C_T &= \text{Price of } C \text{ at } t = T.
\end{aligned}$$

The price of C at time T is a random variable depending on the state, with values given by the m-vector:

$$\vec{C}_T = \begin{bmatrix} C_T^1 \\ C_T^2 \\ \vdots \\ C_T^m \end{bmatrix}$$

The holder of contingent claim C realizes the payoff at time T given by $\vec{C}_T$: the payoff of C, contingent on the state ω_i is C_T^i.

A critical issue for a contingent claim is whether the financial security can be hedged. With a single-period financial model $\mathcal{M} = (\Omega_T, \mathcal{A})$, asset pricing theory provides a framework for analyzing the following questions:

- Can the contingent claim security be replicated by a portfolio of the n assets? If so the m-vector of state-dependent payoffs $\vec{C}_T$ are equal to those of the portfolio.
- If the contingent claim security can be replicated by a portfolio, is the portfolio unique?
- Does the model $\mathcal{M}$ determine C_0, the price of the contingent claim security at $t = 0$?

Basic linear algebra theory for solving systems of linear equations resolves the first two questions. Suppose $\vec{C}_T \in R^m$ specifies the state-dependent payoff/value of a contingent claim C. A portfolio $\vec{\pi} \in R^n$ would replicate the payoffs if:

$$\begin{aligned} \vec{V}(\vec{\pi}) &= \vec{C}_T \\ \text{i.e.,} \quad A\vec{\pi} &= \vec{C}_T \end{aligned}$$

- Such a replicating portfolio $\vec{\pi}$ exists if $\vec{C}_T \in colSpace(A)$.
- If it exists, the replicating portfolio $\vec{\pi}$ is unique if the column-vectors of A are linearly independent.

We can express these properties as a theorem:

Theorem: For the market model $\mathcal{M}$:

1. A replicating portfolio $\vec{\pi}$ exists for any contingent claim C with state-dependent payoffs $\vec{C}_T \in R^m$, if and only if A, the $(m \times n)$ price matrix of the n assets in the m states, has full row rank equal to m.
2. The replicating portfolio $\vec{\pi}$ in (a) is unique if the column rank of A is also m, i.e., there market has m assets whose state-dependent price vectors are linearly independent.

For the third question, we need the concepts of **arbitrage** and **arbitrage-free market models** which we introduce in the next section.

2.4 Arbitrage-Free Market Models and Pricing Measures

An important concept in asset pricing is **arbitrage**: is it possible to implement a trading strategy in a market model which guarantees a profit?

Definition: Arbitrage In a single-period market model $\mathcal{M} = (\Omega_T, \mathcal{A})$ a portfolio $\vec{\pi} = (\pi_1, \ldots, \pi_n)$ is an arbitrage portfolio if the net gain/loss from buying the portfolio at $t = 0$ and selling the portfolio at $t = T$ satisfies:

$$V_T(\vec{\pi}) \geq 0$$

for all states ω_i with strict inequality for at least one i, and

$$V_0(\vec{\pi}) \leq 0.$$

This definition applies a weak definition of *guaranteeing* a profit in that the trading strategy is guaranteed to never incur a loss and to incur a profit in at least one state ω_i.

As explained below, The absence of arbitrage in a single-period market model depends critically on the existence of a *Pricing Measure* which we now define.

Definition: Pricing Measure A probability measure Q on Ω_T given by

$$Q(\omega_i) = q_i > 0,\ i = 1, \ldots, m, \text{ and } \sum_{i=1}^{m} q_i = 1.$$

is a *Pricing Measure* if asset prices can be expressed as:

$$P_0^j = \alpha E^Q[P_T^j] = \alpha \sum_{i=1}^{m} q_i P_T^j(\omega_i), j = 1, \ldots, n,$$

where $\alpha > 0$ is a discount factor applied to every asset.

This state-by-state expression can be written for all states in row-vector-matrix form as:

$$A_0 = \alpha \cdot \vec{q}^T A,$$

which states that the time $t = 0$ prices are the probability-weighted state-dependent outcomes (using probabilities in $\vec{q}$), discounted by the constant factor $\alpha > 0$.

With this definition, we can now state:

Fundamental Theorem of Asset Pricing (Part 1) In a discrete, single-period economy, if all states/scenarios in Ω_T are possible, then

1. There is no arbitrage if and only if there is a pricing measure for which all scenarios are possible.

See Albanese and Campolieti (2006, section 1.1) for a detailed proof of this theorem.

With this theorem we can determine whether a single-period market model $\mathcal{M} = (\Omega_T, \mathcal{A})$ is arbitrage free:

- Consider solving the system of equations:

  $$A_0 = \vec{\psi}^T A,$$

  where $\vec{\psi} = (\psi_1, \ldots, \psi_m)$.

- Suppose a solution vector $\vec{\psi}$ satisfies $\psi_i > 0$ for all i.
- Then a pricing measure Q can be defined with
  $q_i = \psi_i / \sum_{k=1}^{m} \psi_k$, and
  $\alpha = \sum_{k=1}^{m} \psi_k$
- By the theorem, there is no arbitrage in the market model $\mathcal{M}$.

An important concept in asset pricing theory is whether a financial model is *complete.*

Definition: A single-period financial model $\mathcal{M} = (\Omega_T, \mathcal{A})$ is *complete* if any contingent claim C can be replicated by a portfolio $\vec{\pi} \in R^n$. That is, for any

$$\vec{C}_T \in R^m,$$

the $(m \times 1)$ vector of state-dependent payoffs for the contingent claim C, there exists a portfolio $\vec{\pi} \in R^n$ of the n-assets which replicates the time-T payoffs:

$$A\vec{\pi} = \vec{C}_T.$$

The second part of the fundamental theorem details the condition for a market model to be complete.

Fundamental Theorem of Asset Pricing (Part 2) In a discrete, single-period economy, if all states/scenarios in Ω_T are possible, then

- The financial model is complete with no arbitrage if and only if the pricing measure is unique.

From the discussion of Part 1 of the theorem, the pricing measure is unique if the solution for $\vec{\psi}$ is unique. From linear algebra theory, the solution to the system of equations

$$A^T\vec{\psi} = A_0^T$$

is unique only if A_0 is in the rowspace of A, and the row-rank of A is m. This implies that the number of assets in $\mathcal{M}$ is at least $n = m$. If $n > m$, then the all n assets j have payoffs and costs that are replicated, i.e., linearly dependent, on m assets whose time-T payoffs are linearly independent and time-0 costs all are positive.

When a market model is not arbitrage free, this theorem motivates a simple approach to identifying arbitrage opportunities. This approach uses the concept of *Arrow-Debreu Securities* which have the following definition:

Definition: Arrow-Debreu Security In a market model $\mathcal{M} = (\Omega_T, \mathcal{A})$, an Arrow-Debreu Security for state $\omega_i \in \Omega_T$ is the contingent claim $E^{(i)}$ which has time T payoff equal to \$1 if $\omega = \omega_i$ and \$0 otherwise.

For a market model which is not arbitrage free, the following approach identifies Arrow-Debreu securities which can be replicated if the market is complete which have zero or negative cost. Every portfolio replicating an Arrow-Debreu security is an arbitrage opportunity in the market model.

- Let $\vec{\psi}$ be a solution to $A_0 = \vec{\psi}^T A$.
- If the market is complete and an arbitrage opportunity exists, it must be that at least one $\psi_i \leq 0$, else $\vec{\psi}$ would define a pricing measure and discount factor (as detailed above).
- For an index i with $\psi_i \leq 0$, define $\vec{\pi}$ to be the portfolio that replicates the ith Arrow-Debreu Security $E^{(i)}$ which has a payoff of \$1 in state ω_i and \$0 in all other states. The m-vector of time-T payoffs is:

  $$\vec{E}_T^{(i)} = \vec{e}_i,$$

  where $\vec{e}_i$ is the ith column of the order-m identity matrix I_m.

  Constructing the portfolio $\vec{\pi}$ to replicate $E^{(i)}$, the portfolio vector $\vec{\pi}$ must satisfy:

  $$\vec{V}_T(\vec{\pi}) = A\vec{\pi} = \vec{e}_i = \vec{E}_T^{(i)}.$$

- The time $t = 0$ price of the replicating portfolio $\vec{\pi}$ is:

  $$V_0(\vec{\pi}) = A_0\vec{\pi},$$

  and substituting $A_0 = \vec{\psi}^T A$ from above we have

  $$\begin{aligned} V_0(\vec{\pi}) &= [\vec{\psi}^T A]\vec{\pi} = \vec{\psi}^T [A\vec{\pi}] \\ &= \vec{\psi}^T [\vec{e}_i] \\ &= \psi_i \leq 0. \end{aligned}$$

  Thus, the Arrow-Debreu security for state ω_i is an arbitrage opportunity.

3 Visualizing Assets/Portfolios with ggplot2

3.1 Simple Two-Asset Portfolios

Consider the two-asset, two-state model $\mathcal{M}$ where asset 1 is the bond B and asset 2 is the stock S:

the special case of the simple economy with

$B_0 = 100$ and $B_T = 103$.

$S_0 = 100$ and $S_T^u = 130$, $S_T^d = 90$.

The space of vector payoffs at $t = T$ of portfolios $\vec{\pi} = \begin{bmatrix} \pi_B \\ \pi_S \end{bmatrix}$ are given by

$$\vec{C}_T = \begin{bmatrix} \pi_B B_0 + \pi_S S_T^d \\ \pi_B B_0 + \pi_S S_T^u \end{bmatrix}, \quad \text{subject to constraints on } \vec{\pi}.$$

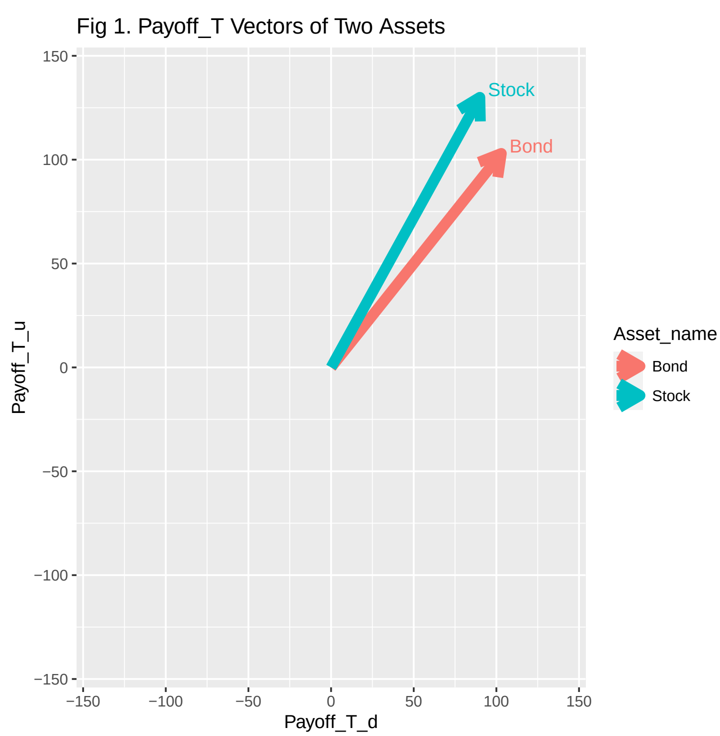
```
> # Define the row matrix of t=0 prices B_0 and S_0
> A0=matrix(c(100,100),nrow=1,ncol=2)
> # Add dimension names for rows and columns
> dimnames(A0)<-list(c("Price_0"), c("Bond","Stock"))
> A0

        Bond Stock
Price_0  100   100

> # Define the matrix of t=T payoffs/prices for B_T and S_T
> #   where row 1 corresponds to state \omega=d and
> #         row 2 corresponds to state \omega=u
> #         column 1 corresponds to the Bond
> #         column 2 corresponds to the Stock
>
> A=cbind(as.matrix(c(103,103)), as.matrix(c(90,130)))
> # Add dimension names
> dimnames(A)<-list(c("Payoff_T_d","Payoff_T_u"),c("Bond","Stock"))
> A

           Bond Stock
Payoff_T_d  103    90
Payoff_T_u  103   130

> # When using ggplot2, data are represented in data frames
> is.data.frame(A)

[1] FALSE

> A.df<-data.frame(A)
> Atranspose.df<-data.frame(t(A))
> Atranspose.df
```

```
      Payoff_T_d Payoff_T_u
Bond         103        103
Stock         90        130
```

```
> # Add Asset_name as a variable/field in Atranspose.df
> Atranspose.df$Asset_name<-dimnames(Atranspose.df)[[1]]
> Atranspose.df
```

```
      Payoff_T_d Payoff_T_u Asset_name
Bond         103        103       Bond
Stock         90        130      Stock
```

```
> library(ggplot2)
> #
> # Figure  AA:  Plot payoff vectors of Two Assets at t=T
> #     Note: the function arrow() specifies the head of the arrows
> g1<-ggplot(data=Atranspose.df,
+            aes(x=Payoff_T_d, y=Payoff_T_u,
+                label=Asset_name, color=Asset_name))
> g1 +geom_segment(aes(x=0,y=0,xend=Payoff_T_d,yend=Payoff_T_u),
+                   size=3,  arrow = arrow(length = unit(0.5, "cm")))+
+     geom_text(aes(label=Asset_name), hjust=0.,vjust=1.,
+           nudge_x=5, nudge_y=7) +xlim(-140,140) + ylim(-140,140) +
+     ggtitle("Fig 1. Payoff_T Vectors of Two Assets")
```



Consider next plot equal to Payoff vectors of collection of portfolios

```
> # Create a data frame data.portfolios
> # with rows corresponding to
> # portfolio cases which contain variables
> #   pi_B     portfolio weights of B
> #   pi_S     portfolio weight of S
> #   Value_0  value/cost of portfolio at t=0
> #   Payoff_T_u payoff of portfolio at t=T for \omega=u
> #   Payoff_T_d payoff of portfolio at t=T for \omega=d
> #   Profit_T_u profit of portfolio at t=T for \omega=u
> #   Profit_T_d profit of portfolio at t=T for \omega=d
> ##
> # pi_B and pi_S correspond to $\pi_B $ and $\pi_S$
> # Consider evaluating portfolios that let these fractions vary
>
>
> # For different cases of sets of portfolios, create a data frame
> # of the portfolio vectors in different rows
> # with weights in respective columns
> list.pi_S=seq(0,1,.1)
> df.pivecs.1<-data.frame(pi_B=1-list.pi_S,pi_S=list.pi_S)
> df.pivecs.1
```

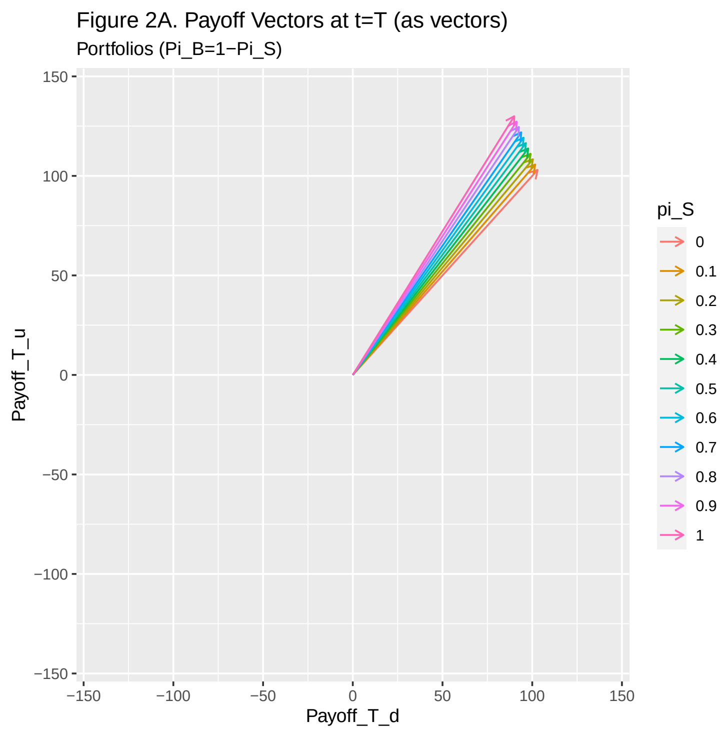
```
   pi_B pi_S
1   1.0  0.0
2   0.9  0.1
3   0.8  0.2
4   0.7  0.3
5   0.6  0.4
6   0.5  0.5
7   0.4  0.6
8   0.3  0.7
9   0.2  0.8
10  0.1  0.9
11  0.0  1.0
```

```
> # Create the data frame data.portfolios
> # with portfolio variables in the columns/fields
>
> for (i in 1:NROW(df.pivecs.1)){
+     pivec=as.vector(df.pivecs.1[i,])
+     Value_0=sum(A0*pivec)
+     Payoff_T_d=sum(A[1,]*pivec)
+     Payoff_T_u=sum(A[2,]*pivec)
+     Profit_T_d=Payoff_T_d-Value_0
+     Profit_T_u=Payoff_T_u-Value_0
```

```
+     new.data.portfolios<-data.frame(pi_B=pivec[1],
+                                  pi_S=pivec[2],
+                                  Value_0=Value_0,
+                                  Payoff_T_d=Payoff_T_d,
+                                  Payoff_T_u=Payoff_T_u,
+                                  Profit_T_d=Profit_T_d,
+                                  Profit_T_u=Profit_T_u)
+       if (i==1){
+         data.portfolios<-new.data.portfolios
+                 }else{
+       data.portfolios<-rbind(data.portfolios, new.data.portfolios)
+                 }
+ }
> data.portfolios.1=data.portfolios
> # load library ggplot2
> library(ggplot2)
```

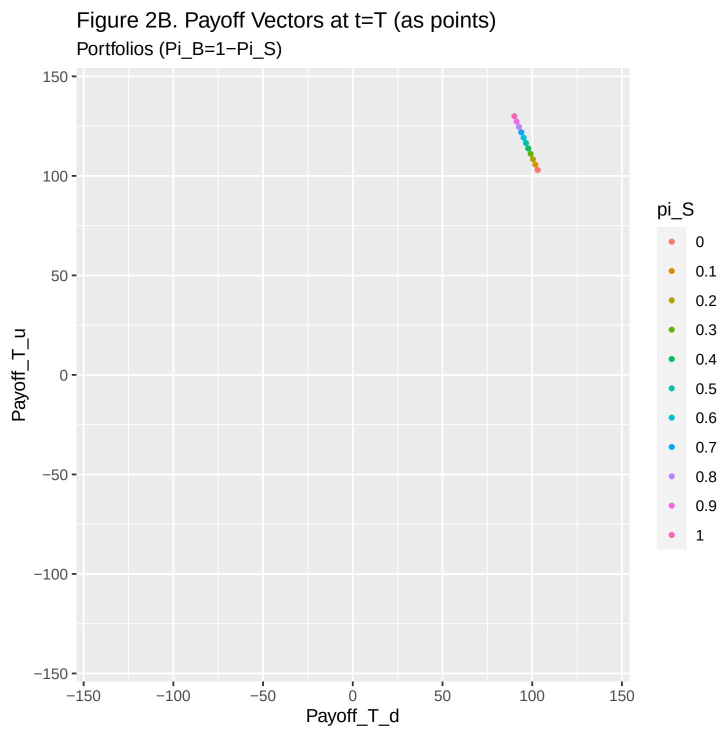
We now use *ggplot* to plot the payoff vectors
$\vec{C}_t = (Payoff^d_T, Payoff^u_T)^T$
for every portfolio in the set. Note that the color of the vectors varies with the $\pi_S = pi_S$, portfolio weight on the stock S.

```
> data.portfolios.1$pi_S <-as.factor(data.portfolios.1$pi_S)
> # Figure 2A:  Plot payoffs as vectors
> g1<-ggplot(data=data.portfolios.1,aes(x=Payoff_T_d,y=Payoff_T_u))
> g2<-g1 + geom_segment(aes(x=0,y=0,xend=Payoff_T_d,yend=Payoff_T_u,
+                        col=pi_S),size=.5,  arrow = arrow(length = unit(0.2, "cm")))+
+   xlim(c(-140,140)) +ylim(c(-140,140))
> g2 + ggtitle("Figure 2A. Payoff Vectors at t=T (as vectors)",
+              subtitle="Portfolios (Pi_B=1-Pi_S)")
>
```



In the next plot, we display the payoff vectors as points. When considering more complex sets of portfolios, the points display is useful because it conveys the same information in a simpler form.

```
> # Figure 2B:  Plot payoffs as points
> g1<-ggplot(data=data.portfolios.1,aes(x=Payoff_T_d,y=Payoff_T_u))
> g2<-g1+ geom_point(aes(col=pi_S),size=1) + xlim(c(-140,140)) +ylim(c(-140,140))
> g2 + ggtitle("Figure 2B. Payoff Vectors at t=T (as points)",
+              subtitle="Portfolios (Pi_B=1-Pi_S)")
>
```



3.2 Portfolios Satisfying Simple Constraints

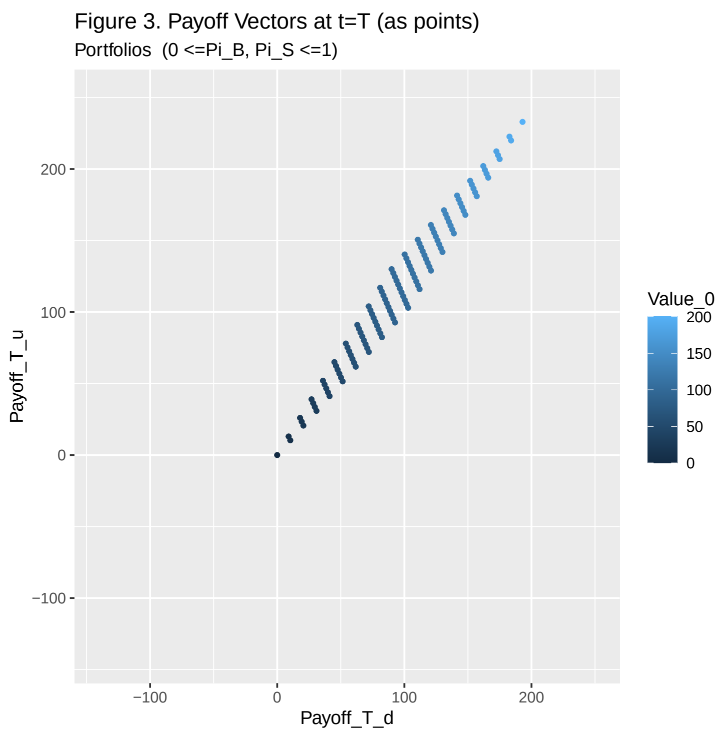
Now, we consider different constraints on the portfolios by defining sets of portfolios satisfying certain constraints

```
> # Consider df.pivecs that lets pi_B and pi_S range between 0 and 1
> list.pi_S=seq(0,1,.1)
> list.pi_B=seq(0,1,.1)
> for (i in 1:length(list.pi_B)){ for (j in 1:length(list.pi_S)){
+   pivec=c(list.pi_B[i], list.pi_S[j])
+   if ((i==1)&&(j==1)){
+     df.pivecs<-data.frame(pi_B=pivec[1], pi_S=pivec[2])} else {
+       df.pivecs<-rbind(df.pivecs,
+                        data.frame(pi_B=pivec[1], pi_S=pivec[2]))}
+ }
+ }
> df.pivecs.2<-df.pivecs
>   head(df.pivecs.2)

  pi_B pi_S
1    0  0.0
2    0  0.1
```

```
3    0  0.2
4    0  0.3
5    0  0.4
6    0  0.5

>   tail(df.pivecs.2)

    pi_B pi_S
116    1  0.5
117    1  0.6
118    1  0.7
119    1  0.8
120    1  0.9
121    1  1.0

> # Write a function that creates data frame with portfolio
> # attributes corresponding to portfolio weights in row of input
> # data frame
> fcn.data.portfolios<-function(df.pivecs){
+   for (i in 1:NROW(df.pivecs)){
+     pivec=as.vector(df.pivecs[i,])
+     Value_0=sum(A0*pivec)
+     Payoff_T_d=sum(A[1,]*pivec)
+     Payoff_T_u=sum(A[2,]*pivec)
+     Profit_T_d=Payoff_T_d-Value_0
+     Profit_T_u=Payoff_T_u-Value_0
+     new.data.portfolios<-data.frame(pi_B=pivec[1],
+                                  pi_S=pivec[2],
+                                  Value_0=Value_0,
+                                  Payoff_T_d=Payoff_T_d,
+                                  Payoff_T_u=Payoff_T_u,
+                                  Profit_T_d=Profit_T_d,
+                                  Profit_T_u=Profit_T_u)
+       if (i==1){
+         data.portfolios<-new.data.portfolios
+                }else{
+       data.portfolios<-rbind(data.portfolios, new.data.portfolios)
+                }
+ }
+ return(data.portfolios)
+ }
>
```

We use the new function and the data frame of portfolio vectors to create a new data frame of portfolio attributes for the set of portfolios. Note that the initial cost of the portfolio is a portfolio variable used to color the points. The greater the payoffs, the higher the initial cost.

```
> data.portfolios.2<-fcn.data.portfolios(df.pivecs.2)
> # Figure 3:  Plot payoffs as points for data.portfolios.2
> g1<-ggplot(data=data.portfolios.2,aes(x=Payoff_T_d,y=Payoff_T_u))
> g2<-g1+ geom_point(aes(col=Value_0),size=1) + xlim(c(-140,250)) +ylim(c(-140,250))
> # Note, the xlim and ylim values are set manually
> g2 + ggtitle("Figure 3. Payoff Vectors at t=T (as points)",
+              subtitle="Portfolios  (0 <=Pi_B, Pi_S <=1)")
```

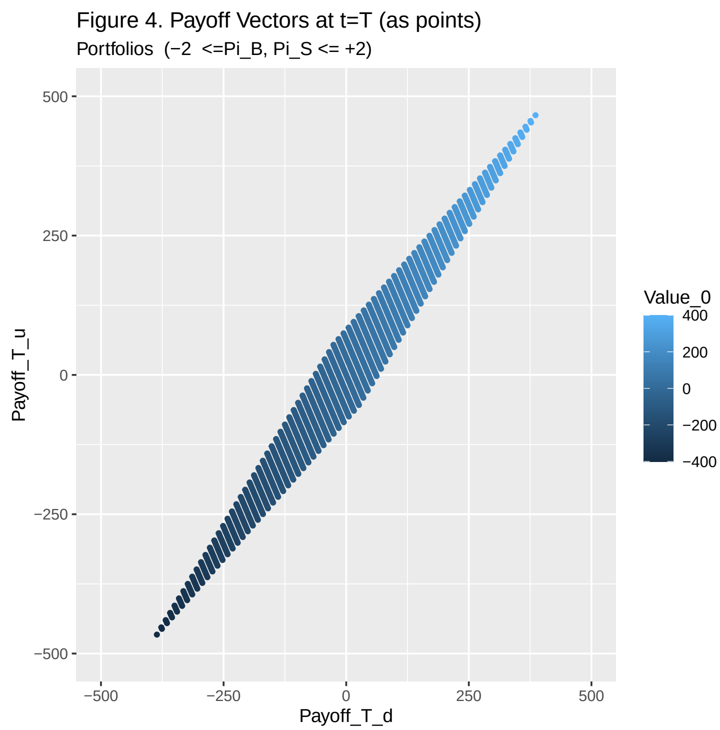


3.3 Portfolios Allowing Shorting

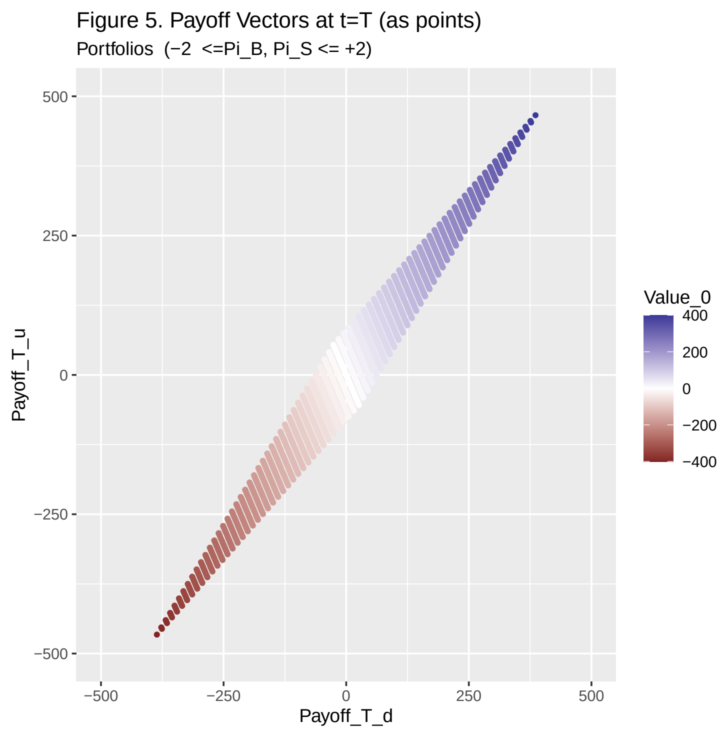
Now, we consider portfolios that allow shorting of either the Bond B or the stock S.

```
> # Consider df.pivecs that let's pi_B and pi_S range between -2 and 2
> list.pi_S=seq(-2,2,.1)
> list.pi_B=seq(-2,2,.1)
> for (i in 1:length(list.pi_B)){ for (j in 1:length(list.pi_S)){
+   pivec=c(list.pi_B[i], list.pi_S[j])
+   if ((i==1)&&(j==1)){
+     df.pivecs<-data.frame(pi_B=pivec[1], pi_S=pivec[2])} else {
+       df.pivecs<-rbind(df.pivecs,
+                        data.frame(pi_B=pivec[1], pi_S=pivec[2]))}
```

```
+ }
+ }
> df.pivecs.3<-df.pivecs
>   head(df.pivecs.3)

  pi_B pi_S
1   -2 -2.0
2   -2 -1.9
3   -2 -1.8
4   -2 -1.7
5   -2 -1.6
6   -2 -1.5

>   tail(df.pivecs.3)

     pi_B pi_S
1676    2  1.5
1677    2  1.6
1678    2  1.7
1679    2  1.8
1680    2  1.9
1681    2  2.0

> data.portfolios.3<-fcn.data.portfolios(df.pivecs.3)
> # Figure 4:  Plot payoffs as points for data.portfolios.3
> g1<-ggplot(data=data.portfolios.3,aes(x=Payoff_T_d,y=Payoff_T_u))
> g2<-g1+ geom_point(aes(col=Value_0),size=1) + xlim(c(-500,500)) +ylim(c(-500,500))
> # Note, the xlim and ylim values are set manually
> g2 + ggtitle("Figure 4. Payoff Vectors at t=T (as points)",
+              subtitle="Portfolios  (-2  <=Pi_B, Pi_S <= +2)")
```




```
> # Redo the prevous plot with gradient2
> # Figure 5:  Plot payoffs as points for data.portfolios.3
> g1<-ggplot(data=data.portfolios.3,aes(x=Payoff_T_d,y=Payoff_T_u))
> g2<-g1+ geom_point(aes(col=Value_0),size=1) + xlim(c(-500,500)) +ylim(c(-500,500)) + scale
> # Note, the xlim and ylim values are set manually
> g2 + ggtitle("Figure 5. Payoff Vectors at t=T (as points)",
+              subtitle="Portfolios  (-2  <=Pi_B, Pi_S <= +2)")
>
```



This graph displays the time $t = T$ payoffs of all portfolios

$$\vec{\pi} = (\pi_B, \pi_S) \in \mathcal{S} = \{portfolios\ \vec{\pi}\ \ satisfying\ \ constraints\}$$ as points in the space

$$(x, y) = Payoff_T(\omega_T = u \mid \vec{\pi}) \text{ vs } Payoff_T(\omega_T = d \mid \vec{\pi})$$

The color changes when $Value_0$ (cost at t=0) varies from negative (shades of red) to positive (shades of blue) costs. Note that the zero-cost portfolios (in white) do not admit positive payoffs in both states (u and d); i.e., there is no arbitrage.

4 References

- *Advanced Derivatives Pricing and Risk Management,* Claudio Albanese and Giuseppe Campolieti, Elsevier Academic Press, 2006.