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Probability Theory For Asset Pricing

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Single Risky Asset

Analytic Framework

- One-period model
  - Time t_0: time of transaction
  - Time t_1: end-of-period.
- Prior to t_0 market agent receives an endowment of
  - Q_a shares of risky asset ("a")
  - Q_f units (dollars) of risk-free asset
- At time t_0
  - Risk-free asset costs \$1/unit
  - Risky asset costs p_a/share.
  - Market agent wealth at t_0

    $$w_0 = Q_a p_a + Q_f$$
- At time t_1
  - Risk-free asset pays $(1 + r_f)$ per unit
  - Risky asset pays $\tilde{F}$/share, where $\tilde{F} \sim Normal(\mu, \sigma^2)$.

Analytic Framework (continued)

- Market agent
  - Buys X units of risky asset a at time t_0.
    (sells if $X < 0$)
  - End-of-period wealth at time t_1 is
    $$\tilde{w} = (Q_a + X)\tilde{F} + (Q_f - Xp_a)(1 + r_f)$$
  - Utility of end-of-period wealth: $U(\tilde{w})$,
    (utility function $U(\cdot)$: U' and U'' exist)
- Agent's optimal choice of X
  - Choose X to maximize Expected Utility
    $$\max_X E[U(\tilde{w})]$$
  - Consider general solution satisfying the first-order condition
    $$\frac{\partial E[U(\tilde{w})]}{\partial X} = 0$$
  - Solve for equilibrium price

Stein's Lemma.

- Suppose that Y and Y^* are jointly normally distributed with $Y \sim Normal(\mu, \sigma^2)$.
- Let g be a continuous, differentiable function for which
$$E[g(Y)(Y-\mu)] \text{ and } E[g'(Y)] \text{ exist.}$$

Then:
$$\begin{aligned} E[g(Y)(Y-\mu)] &= \sigma^2 E[g'(Y)] \text{ and} \\ Cov[g(Y), Y^*] &= E[g'(Y)]Cov(Y, Y^*). \end{aligned}$$

Proof: (Integration by parts)

Wealth at time t_1

$$\tilde{w} = (Q_a + X)\tilde{F} + (Q_f - Xp_a)(1 + r_f)$$

Note: If $\tilde{F} \sim Normal(\mu, \sigma^2)$, then $\tilde{w}$ also normal

First-Order Condition

$$\begin{aligned} 0 &= \frac{\partial E[U(\tilde{w}) \mid X]}{\partial X} \\ &= E[U'(\tilde{w})(\tilde{F} - p_a(1 + r_f))] \\ &= Cov[U'(\tilde{w}), (\tilde{F} - p_a(1 + r_f))] + E[U'(\tilde{w})]E[\tilde{F} - p_a(1 + r_f)] \end{aligned}$$

By Stein's Lemma

$$\begin{aligned} Cov[U'(\tilde{w}), (\tilde{F} - p_a(1 + r_f)) &= E[U''(\tilde{w})]Cov[\tilde{w}, (\tilde{F} - p_a(1 + r_f)] \\ &= E[U''(\tilde{w})]Cov[\tilde{w}, \tilde{F}] \end{aligned}$$

$$\Longrightarrow 0 = E[U''(\tilde{w})]Cov[\tilde{w}, \tilde{F}] + E[U'(\tilde{w})]E[\tilde{F} - p_a(1 + r_f)]$$

Substituting $Cov[\tilde{w}, \tilde{F}] = (Q_a + X)Var(\tilde{F})$,

$$\Longrightarrow 0 = E[U''(\tilde{w})](Q_a + X)Var(\tilde{F}) + E[U'(\tilde{w})]E[\tilde{F} - p_a(1 + r_f)]$$

$$\Longrightarrow p_a = \frac{1}{1+r_f}[E(\tilde{F}) + \frac{E[U''(\tilde{w})]}{E[U'(\tilde{w})]}(Q_a + X)Var(\tilde{F})]$$

Equilibrium Price:

$$p_a = \frac{1}{1+r_f}\left[E(\tilde{F}) + \frac{E[U''(\tilde{w})]}{E[U'(\tilde{w})]}(Q_a + X)Var(\tilde{F})\right]$$

Constant Absolute Risk Aversion (CARA) Utility

$$U(\tilde{w}) = (constant) - e^{-A\tilde{w}}$$

Note:

$$\begin{aligned} U'(\tilde{w}) &= AU(\tilde{w}) \\ U''(\tilde{w}) &= -A^2U(\tilde{w}) \end{aligned}$$

$$\implies \quad p_a = \frac{1}{1+r_f}\left[E(\tilde{F}) - A(Q_a + X)Var(\tilde{F})\right]$$

- Price equals discounted expected cash flow minus risk adjustment
- Risk adjustment in price formula proportional to
    - Risk aversion coefficient (A)
    - Shares of risky asset ($Q_A + X$).
    - Variance in price $Var(\tilde{F})$

From equilibrium price

$$p_a = \frac{1}{1+r_f}\left[E(\tilde{F}) + \frac{E[U''(\tilde{w})]}{E[U'(\tilde{w})]}(Q_a + X)Var(\tilde{F})\right]$$

Expected Return of Risky Asset

$$\begin{aligned} E[\tilde{r}] &= \frac{E(\tilde{F}) - p_a}{p_a} \\ &= r_f - \frac{E[U''(\tilde{w})]}{E[U'(\tilde{w})]}(Q_a + X)Var(\tilde{F})/p_a] \\ &= r_f + A(Q_a + X)Var(\tilde{F})/p_a] \text{ for CARA utility} \end{aligned}$$

Note: Expected return on risky asset equals sum of

- Risk-free rate
- Risk premium, which depends on
  - risk aversion (A)
  - shares of risky asset ($Q_a + X$)
  - asset variance: $Var(\tilde{F})$

Analytic Framework

- One-period model
  - Time t_0: time of transaction
  - Time t_1: end-of-period.
- Prior to t_0 market agent receives an endowment of
  - $\vec{Q} = (Q_1, \ldots, Q_n)^\top$ shares of risky assets $(a_1, a_2, \ldots, a_n)$
  - Q_f units (dollars) of risk-free asset
- At time t_0
  - Risk-free asset costs \$1/unit
  - Risky asset cost/share: $\vec{p} = (p_{a_1}, p_{a_2}, \ldots, p_{a_n})^\top$
  - Market agent wealth at t_0:
    $$w_0 = \vec{Q}^\top \vec{p} + Q_f$$
- At time t_1
  - Risk-free asset pays $(1 + r_f)$ per unit
  - Risky assets pay $\vec{F} = [\tilde{F}_1, \ldots, \tilde{F}_n]^\top$ per share.

Multiple Risky Assets

Analytic Framework (continued)

- Market agent
  - Buys $\vec{X} = [X_1, \ldots, X_n]^\top$ units of risky assets at time t_0. (sells with $X_i < 0$)
  - End-of-period wealth at time t_1 is
    $$\tilde{w} = (\vec{Q} + \vec{X})^\top \vec{F} + (Q_f - \vec{X}^\top \vec{p})(1 + r_f)$$
  - Utility of end-of-period wealth: $U(\tilde{w})$, (utility function $U(\cdot)$: U' and U'' exist)
- Agent's optimal choice of X
  - Choose X to maximize Expected Utility
    $$\max_X E[U(\tilde{w})]$$
  - Consider general solution satisfying the first-order condition
    $$\frac{\partial E[U(\tilde{w})]}{\partial X} = 0$$
  - Solve for equilibrium price

Multiple Risky Assets

Wealth at t_1: $\tilde{w} = (\vec{Q} + \vec{X})^\top \tilde{F} + (Q_f - X^\top \vec{p})(1 + r_f)$

Claim: If $\tilde{\vec{F}}$ jointly Normal then $\tilde{w}$ also normal.

First-Order Conditions $(i = 1, \ldots, n)$

$$\begin{aligned} 0 &= \frac{\partial E[U(\tilde{w})]}{\partial X_i}, \\ &= E[U'(\tilde{w})(\tilde{F}_i - p_{a_i}(1 + r_f))] \\ &= Cov[U'(\tilde{w}), (\tilde{F}_i - p_{a_i}(1 + r_f))] + E[U'(\tilde{w})]E[\tilde{F}_i - p_{a_i}(1 + r_f)] \end{aligned}$$

Again by Stein's Lemma:

$$\begin{aligned} Cov[U'(\tilde{w}), (\tilde{F}_i - p_{a_i}(1 + r_f))] &= E[U''(\tilde{w})]Cov[\tilde{w}, \tilde{F}_i - p_{a_i}(1 + r_f)] \\ &= E[U''(\tilde{w})]Cov[\tilde{w}, \tilde{F}_i] \end{aligned}$$

$$\Longrightarrow 0 = E[U''(\tilde{w})]Cov[\tilde{w}, \tilde{F}_i] + E[U'(\tilde{w})]E[\tilde{F}_i - p_{a_i}(1 + r_f)]$$

$$\Longrightarrow p_{a_i} = \frac{1}{1 + r_f}\left[E(\tilde{F}_i) + \frac{E[U''(\tilde{w})]}{E[U'(\tilde{w})]}Cov[\tilde{w}, \tilde{F}_i]\right]$$

Note: $Cov[\tilde{w}, \tilde{F}_i] = \left[Var(\tilde{F})(\vec{Q} + \vec{X})\right]_i,$

Equilibrium Prices of the n Assets:

$$p_{a_i} = \frac{1}{1+r_f}\left[E(\tilde{F}_i) + \frac{E[U''(\tilde{w})]}{E[U'(\tilde{w})]}Cov[\tilde{w}, \tilde{F}_i]\right]$$

Returns of Risky Assets

$$\tilde{r}_i = \frac{\tilde{F}_i - p_{a_i}}{p_{a_i}}$$

Expected Returns of Risky Assets

$$\begin{aligned} E[\tilde{r}_i] &= \frac{E(\tilde{F}_i) - p_{a_i}}{p_{a_i}} \\ &= r_f - \frac{E[U''(\tilde{w})]}{E[U'(\tilde{w})]}Cov[\tilde{w}, \tilde{r}_i] \\ &= r_f + ACov(\tilde{w}, \tilde{r}_i) \quad \text{(for CARA utility)} \end{aligned}$$

Note: Expected return on risky asset equals sum of

- Risk-free rate
- Risk premium (depending on risk aversion, nonzero asset covariances with asset a_i, and shares of those assets)

Capital Asset Pricing Model (CAPM)

Market Portfolio

- Assume zero initial endowment in risky assets: $\vec{Q} = \vec{0}$
- Consider $\vec{X} = (X_1, \ldots, X_n)^\top$ the equilibrium investment in the market portfolio with cash flow $\tilde{F}_m$ at t_1 and price p_m at t_0

$$\tilde{F}_m = \textstyle\sum_{i=1}^n X_i \tilde{F}_i \qquad p_m = \sum_{i=1}^n X_i p_{a_i}$$

Return on Market Portfolio

$$\begin{aligned} \tilde{r}_m &= \frac{\tilde{F}_m - p_m}{p_m} = \frac{\sum_{j=1}^n X_i(\tilde{F}_i - p_{a_i})}{p_m} \\ &= \textstyle\sum_{i=1}^n \left[\frac{X_i p_{a_i}}{p_m}\right] \frac{\tilde{F}_i - p_{a_i}}{p_{a_i}} = \displaystyle\sum_{i=1}^n \mu_i \tilde{r}_i \end{aligned}$$

Expected Return of Market Portfolio

$$\begin{aligned} E[\tilde{r}_m] &= \textstyle\sum_{i=1}^n \mu_i E[\tilde{r}_i] \\ &= \textstyle\sum_{i=1}^n \mu_i (r_f - \left[\frac{E[U''(\tilde{w})]}{E[U'(\tilde{w})]}\right] Cov[\tilde{w}, \tilde{r}_i]) \end{aligned}$$

CAPM Expected Returns

Expected Return of Market Portfolio

$$
\begin{aligned}
E[\tilde{r}_m] &= \sum_{i=1}^{n} \mu_i E[\tilde{r}_i] \\
&= \sum_{i=1}^{n} \mu_i \left(r_f - \left[\frac{E[U''(\tilde{w})]}{E[U'(\tilde{w})]}\right] Cov[\tilde{w}, \tilde{r}_i]\right) \\
&= r_f + \left[-\frac{E[U''(\tilde{w})]}{E[U'(\tilde{w})]}\right] \sum_{i=1}^{n} \mu_i Cov[\tilde{w}, \tilde{r}_i] \\
&= r_f + \left[-\frac{E[U''(\tilde{w})]}{E[U'(\tilde{w})]}\right] Cov\left[\tilde{w}, \sum_{i=1}^{n} \mu_i \tilde{r}_i\right] \\
&= r_f + \left[-\frac{E[U''(\tilde{w})]}{E[U'(\tilde{w})]}\right] Cov[\tilde{w}, \tilde{r}_m]
\end{aligned}
$$

$$
\Longrightarrow \left[-\frac{E[U''(\tilde{w})]}{E[U'(\tilde{w})]}\right] = \frac{E[\tilde{r}_m] - r_f}{Cov[\tilde{w}, \tilde{r}_m]} = \frac{E[\tilde{r}_m] - r_f}{p_m Cov[\tilde{r}_m, \tilde{r}_m]} = \frac{E[\tilde{r}_m] - r_f}{p_m Var[\tilde{r}_m]}
$$

CAPM Expected Returns

Expected Return for Individual Asset

$$
\begin{aligned}
E[\tilde{r}_i] - r_f &= \left[-\frac{E[U''(\tilde{w})]}{E[U'(\tilde{w})]} \right] Cov[\tilde{w}, \tilde{r}_i] = \left[\frac{E[\tilde{r}_m] - r_f}{p_m Var[\tilde{r}_m]} \right] p_m Cov[\tilde{r}_m, \tilde{r}_i] \\
&= \frac{Cov[\tilde{r}_m, \tilde{r}_i]}{Var[\tilde{r}_m]} \left[E[\tilde{r}_m] - r_f \right] \\
&= \beta_i \left[E[\tilde{r}_m] - r_f \right]
\end{aligned}
$$

Security Market Line

- $R_i = E[\tilde{r}_i]$
- $R_m = E[\tilde{r}_m]$
- $r_f =$ risk-free rate

$$
\boxed{R_i = r_f + \beta_i (R_m - r_f)}
$$

CAPM Equilibrium Prices

Equilibrium Price of Individual Assets

$$\begin{aligned} E[\tilde{r}_i\ ] &= \frac{E[\tilde{F}_i] - p_{a_i}}{p_{a_i}} \\ \implies 1 + E[\tilde{r}_i\ ] &= \frac{E[\tilde{F}_i]}{p_{a_i}} \\ \implies p_{a_i} &= \frac{E[\tilde{F}_i]}{1 + E[\tilde{r}_i\ ]} \end{aligned}$$

$$p_{a_i} = \frac{E[\tilde{F}_i]}{1 + r_f + \beta_i[E[\tilde{r}_m] - r_f]}$$

References

- *The Microstrucrure of Financial Markets,* Frank de Jong and Barbara Rindi, Cambridge University Press, 2009.
- *The Econometrics of Financial Markets,* John Y. Campbell, Andrew W. Lo, and A. Craig MacKinlay, Princeton University Press, 1997.
- *Foundations of Finance,* Eugene F. Fama, Basic Books, Inc. New York, 1976.
- Lintner, John (1965). "The Valuation of Risk Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budgets." *Review of Economics and Statistics 47 (February):* 13-37.
- Sharpe, W. (1964). "Capital Asset Prices: A Theory of Market Equilibrium under Conditions of Risk," Journal of Finance, 19: 425-442.