Stochastic Processes I

MIT 18.642

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Martingale Definition

- $\{X_n, 1 \leq n < \infty\}$, a sequence of random variables
- $M_n = f_n(X_1, \ldots, X_n) : R^n \to R$

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$\{M_n, n = 1, 2, \ldots\}$ is a Martingale if

- $E[M_n \mid X_1, X_2, \ldots, X_{n-1}] = M_{n-1}$, for all $n \geq 1$
- $E[|M_n|] < \infty$ (M_0 a finite constant) , for all $n \geq 1$

Examples of Martingales

Example 1

- X_n independent random variables
- $E(X_n) = 0$, for all $n \geq 1$
- $S_n = X_1 + X_2 + \cdots X_n$

$\{S_n, n = 1, 2, \ldots\}$ is a Martingale with respect to $\{X_n, n \geq 1\}$

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$$E[X_i] = 0$$

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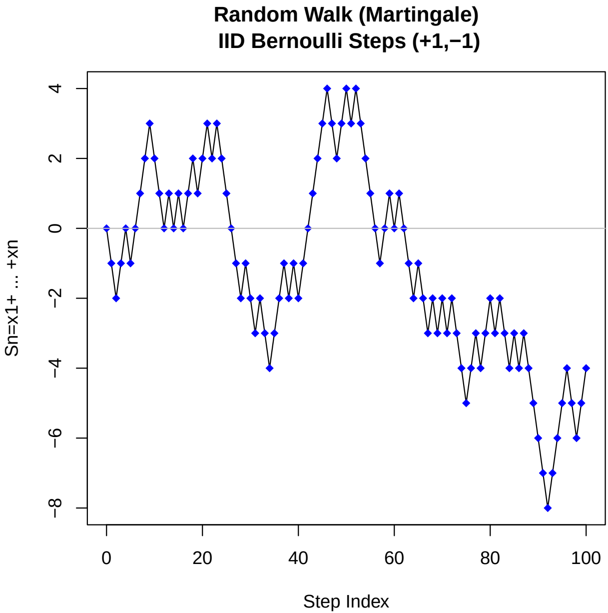
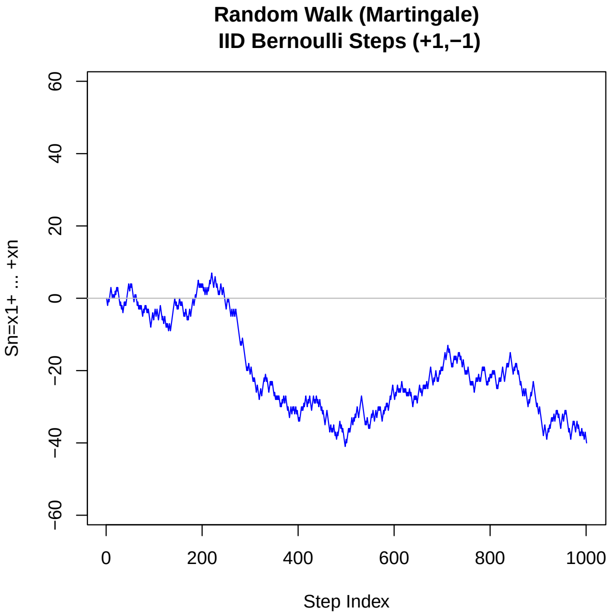
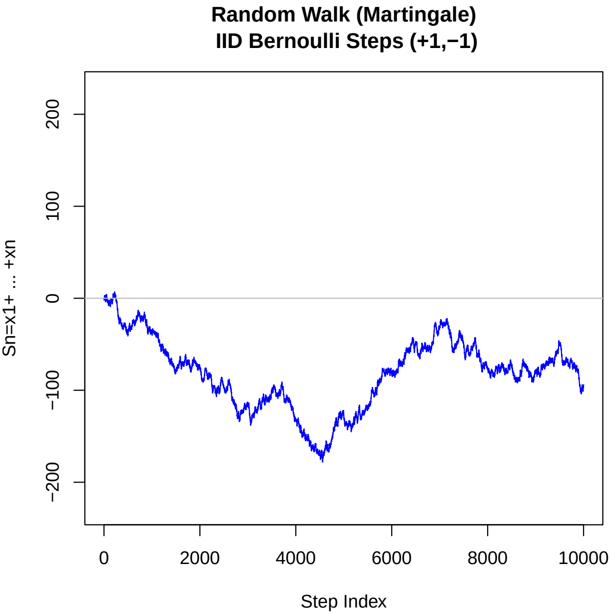
Random Walks




Random Walk (Martingale)
IID Bernoulli Steps (+1,−1)
Sn=x1+ ... +xn
60
40
20
0
−20
−40
−60
0
200
400
600
800
1000
Step Index

Random Walks



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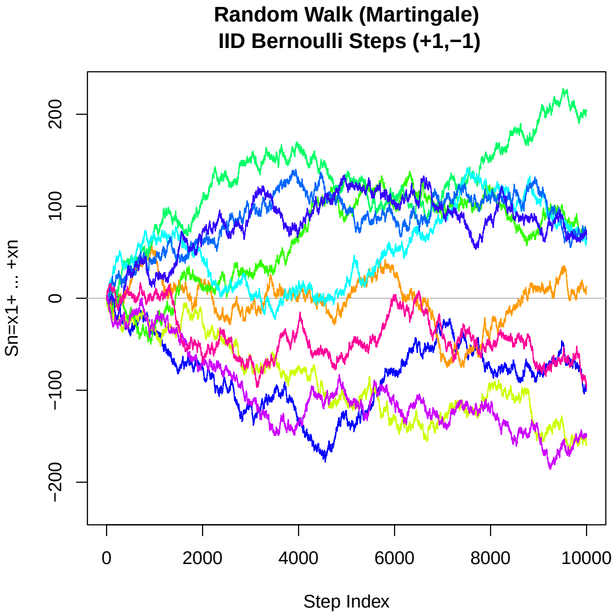

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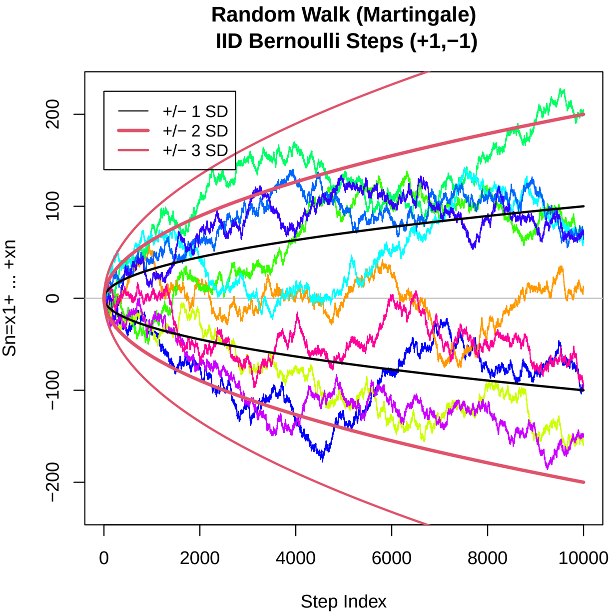


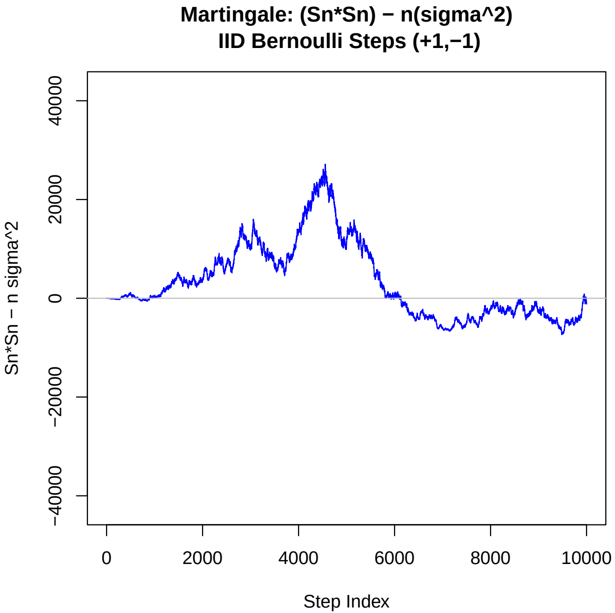
Examples of Martingales

Example 2

- X_n independent random variables
- $E(X_n) = 0$, for all $n \geq 1$
- $Var(X_n) = \sigma^2$, for all $n \geq 1$
- $S_n = X_1 + X_2 + \cdots X_n$
- $M_n = S_n^2 - n\sigma^2$

$\{M_n, n = 1, 2, \ldots\}$ is a Martingale with respect to $\{X_n, n \geq 1\}$

Martingale: Example 2



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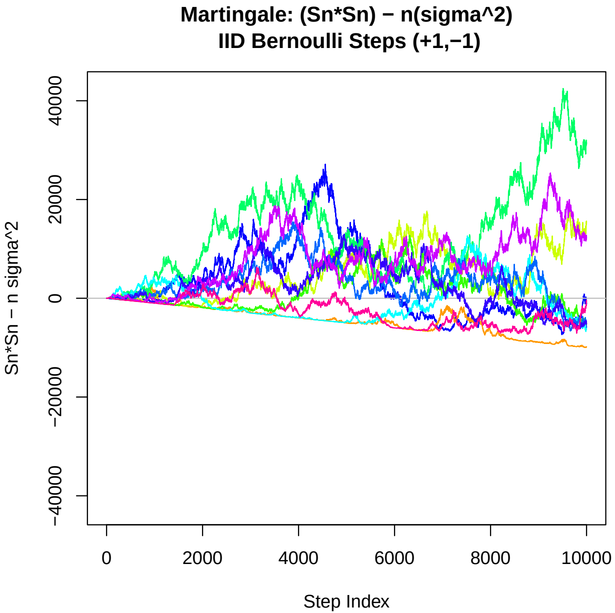

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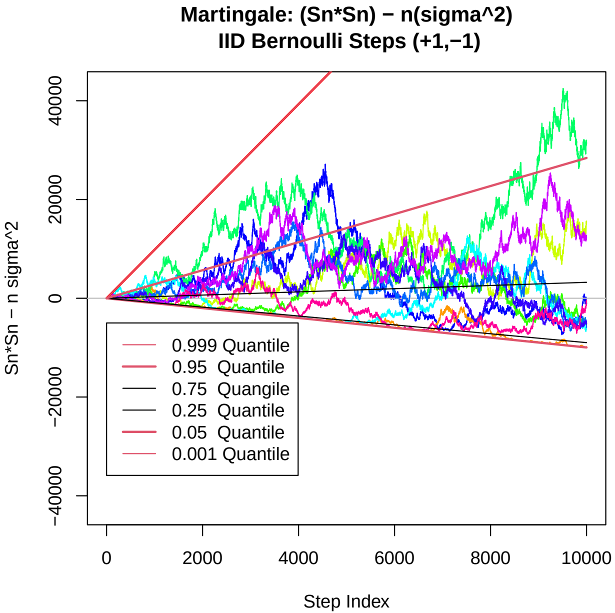


Examples of Martingales

Example 3

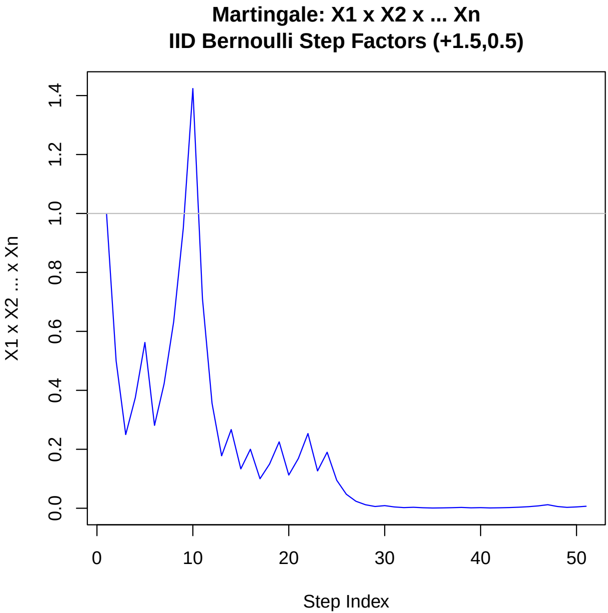
- X_n independent random variables
- $X_n \geq 0$
- $E(X_n) = 1$, for all $n \geq 1$
- $M_n = X_1 \times X_2 \times \cdots \times X_n$
- $M_0 = 1$

$\{M_n, n = 1, 2, \ldots\}$ is a Martingale with respect to $\{X_n, n \geq 1\}$

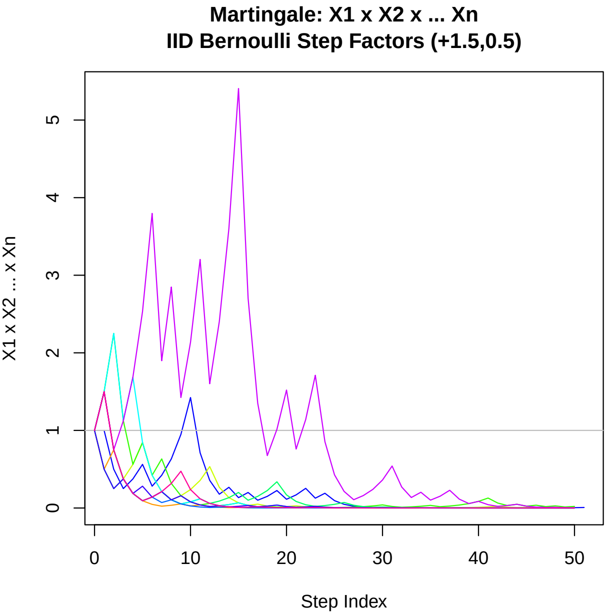
Example: $X_i \sim \mathit{Bernoulli}(.5)$ on $\{.5, 1.5\}$

$$P(X_i = x) = 0.5, \text{ for } x = +1.5, x = +0.5.$$

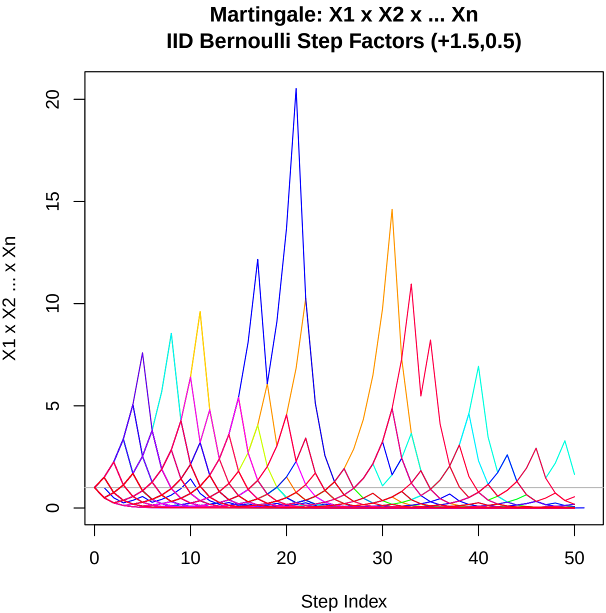
Martingales: Example 3 (1 Path)



Martingales: Example 3 (10 Paths)



Martingales: Example 3 (100 Paths)



Example 4

- Y_n are i.i.d. random variables
- Moment generating function of Y_n is
  $$\phi(\lambda) = E(e^{\lambda Y_n}) < \infty$$
- Define $X_n = e^{\lambda Y_n}/\phi(\lambda)$
  Note: $E[X_n] = \phi(\lambda)/\phi(\lambda) = 1$
- $M_n = exp(\lambda \sum_{i=1}^{n} Y_i)/[\phi(\lambda)]^n$

$\{M_n, n = 1, 2, \ldots\}$ is a Martingale with respect to $\{X_n, n \geq 1\}$
for any fixed λ

Special Case

- There exists λ_0: $\phi(\lambda_0) = 1$
- $M_n = exp(\lambda_0 S_n)/[\phi(\lambda_0)]^n = exp(\lambda_0 S_n)$, where $S_n = \sum_1^n Y_n$

Non-anticipating Random Variables

- $\mathcal{F}_n$: information set on $(X_1, X_2, \ldots, X_n)$
- $E[Z \mid X_1, X_2, \ldots X_n] = E[Z \mid \mathcal{F}_n]$
- $\{\mathcal{F}_n, n = 1, 2, \ldots\}$ is a *Filtration*
- $\mathcal{F}_n$ includes set of all paths up to time n.
- Martingale w.r.t. $\{X_n, n \geq 1\} \equiv$ Martinagale w.r.t. $\{\mathcal{F}_n\}$
- $Y \in \mathcal{F}_n \Leftrightarrow \exists f : Y = f(X_1, X_2, \ldots, X_n)$

Random variables $\{A_n, n \geq 1\}$ are "non-anticipating w.r.t. $\{\mathcal{F}_n\}$"

$$\text{if} \quad \forall \; 1 \leq n < \infty, \; A_n \in \mathcal{F}_{n-1}$$

Martingale Transform Theorem

Definition $\{\tilde{M}_n, n \geq 0\}$ is the Martingale Transform of the martingale $\{M_n, n \geq 0\}$ by $\{A_n\}$, a non-anticipating sequence of random variables if

$$\begin{aligned}\tilde{M}_n &= M_0 + A_1(M_1 - M_0) + A_2(M_2 - M_1) \\ &\quad + \cdots + A_n(M_n - M_{n-1}), \quad \text{for } n \geq 1\end{aligned}$$

Martingale Transform Theorem
If $\{A_n, n \geq 1\}$:

- Bounded random variables
- Non-anticipating w.r.t. $\{\mathcal{F}_n\}$

Then $\{\tilde{M}_n, n \geq 1\}$ is a martingale w.r.t. $\{\mathcal{F}_n\}$

Definition: Stopping Time Random Variable τ

- Stochastic process: $\{X_n, n = 0, 1, \ldots\}$
  with state space $S = \{0, 1, 2, \ldots\}$
- $\{\mathcal{F}_n\} = \{\mathcal{F}_n, n = 0, 1, \ldots\}$:
  $\mathcal{F}_n =$ information set on $(X_0, X_1, \ldots, X_n)$.
- τ is a random variable on $S = \{0, 1, 2, \ldots\} \cup \{\infty\}$
- τ is a stopping time random variable if
  $\{\tau \leq n\} \in \mathcal{F}_n, \forall\ 0 \leq n < \infty$

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For a Stopping Time Random Variable τ

- $\tau = \tau(\omega), \omega \in \Omega$ where
  $\Omega = \{\mathit{all\ possible\ paths\ of}\,(X_1, X_2, \dots)\}$
- For all times n, the event
  $E = \{\omega : \tau(\omega) \le n\}$ is known,
  I.e. $1(\omega \in E)$ is 0 or 1.

  (Either we know the specific time $\tau(\omega) \le n$ or
  we know that $\tau(\omega) > n$.)

Definition: Stopped Process X_τ

- Stochastic process: $\{X_n, n = 0, 1, \ldots\}$
- τ: a stopping time random variable on $\{\mathcal{F}_n\}$.
- For $\{X_n\}$, the stopped process with respect to the stopping time τ is

$$X_\tau = \sum_{n=0}^{\infty} X_n \times \delta_{\tau,n}$$

where

$$\delta_{\tau,n} = \begin{cases} 1 & \text{if } \tau = n \\ 0 & \text{otherwise} \end{cases}$$

Truncated Stopping Times

Definition: Truncated Stopping Time Random Variable $n \wedge \tau$

- τ: a stopping time w.r.t. $\{\mathcal{F}_n\}$
- $n \wedge \tau = min(n, \tau)$

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For a finite value $n < \infty$, the truncated stopping time $n \wedge \tau$ is

measurable w.r.t $\mathcal{F}_m$, for $m \geq n$

not necessarily measurable for $m < n$.

Stopping Time Theorem

Theorem:

If $\{M_n, n \geq 1\}$ is a martingale w.r.t. $\{\mathcal{F}_n\}$
Then $\{M_{n\wedge\tau}, n \geq 1\}$ is also martingale w.r.t. $\{\mathcal{F}_n\}$

Proof:

- Assume $M_0 = 0$, else replace M_n by $M'_n = M_n - M_{n-1}$
- Define $\{A_n, n \geq 1\}$ such that for fixed k
$$A_k = \mathbf{1}(\tau \geq k) = 1 - \mathbf{1}(\tau \leq k-1)$$
$\{A_n, n \geq 1\}$ are non-anticipating w.r.t. $\{\mathcal{F}_n\}$
$$\begin{aligned} \sum_1^n A_k(M_k - M_{k-1}) &= M_\tau \mathbf{1}(\tau \leq k-1) + M_n \mathbf{1}(\tau \geq k) \\ &= M_{n\wedge\tau} \end{aligned}$$
- By the Martingale Transform Theorem
$\{M_{n\wedge\tau}, n \geq 1\}$ is a martingale

Simple Random Walk

- X_n i.i.d. $\quad P(X_n = +1) = P(X_n = -1) = \frac{1}{2}$
- $S_n = \sum_1^n X_i, \quad S_0 = 0$
- Hit Levels: $+A$, and $-B$
- $\tau = min\{n : S_n = +A, \ \ or \ \ S_n = -B\}$

Problem: Solve for $P(\tau = +A)$

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Players 1 and 2 with bank rolls \$A and \$B, continuously bet \$1 with even odds of winning until one loses entire bank roll.

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- $\{S_{n\wedge\tau}\}$ is a martingale by the Stopping Time Theorem

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- $\{S_{n\wedge\tau}\}$ is a martingale by the Stopping Time Theorem
- $E(S_{n\wedge\tau}) = E(S_{0\wedge\tau}) = 0,$ for all n
- $\lim_{n\to\infty} E(S_{n\wedge\tau}) = S_\tau$ w.p. 1 given $P(\tau = \infty) = 0$
- $\Longrightarrow 0 = E(S_\tau)$

Solve for $P(\tau = +A)$

$$S_\tau = +A \cdot \mathbf{1}(S_\tau = +A) - B \cdot \mathbf{1}(S_\tau = -B)$$

Solve for $P(\tau = +A)$

$$
\begin{aligned}
S_\tau &= +A \cdot \mathbf{1}(S_\tau = +A) \;\; - B \cdot \mathbf{1}(S_\tau = -B) \\
0 = E[S_\tau] &= +A \cdot P(S_\tau = +A) \;\; - B \cdot P(S_\tau = -B)
\end{aligned}
$$

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$$\begin{aligned} S_\tau &= +A \cdot \mathbf{1}(S_\tau = +A) \;\; - B \cdot \mathbf{1}(S_\tau = -B) \\ 0 = E[S_\tau] &= +A \cdot P(S_\tau = +A) \;\; - B \cdot P(S_\tau = -B) \end{aligned}$$

$$P(S_\tau = -B) = 1 - P(S_\tau = +A)$$

$$\implies P(S_\tau = +A) = \left(\frac{B}{A+B}\right)$$

Solve for Expected Hitting Time: $E(\tau)$

- $\{M_n\}$ where $M_n = S_n^2 - n$ is a martingale (by Example 2)
- $M_{n\wedge\tau} \leq max(A^2, B^2) + \tau$, bounded r.v.
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- $E(M_\tau) = 0 = E(S_\tau^2 - \tau) = E(S_\tau^2) - E(\tau)$

$\Longrightarrow$

$$\begin{aligned} E(\tau) = E(S_\tau^2) &= P(S_\tau = +A)\cdot A^2 + P(S_\tau = -B)\cdot B^2 \\ &= \left(\frac{B}{A+B}\right)\cdot A^2 + \left(\frac{A}{A+B}\right)\cdot B^2 = AB \end{aligned}$$

Random Walk with Bias

- X_n i.i.d.

$$\begin{aligned} P(X_n = +1) &= p \\ P(X_n = -1) &= q = 1 - p \ \ (0 < p < 1) \end{aligned}$$

- $S_n = \sum_1^n X_i, \quad S_0 = 0$
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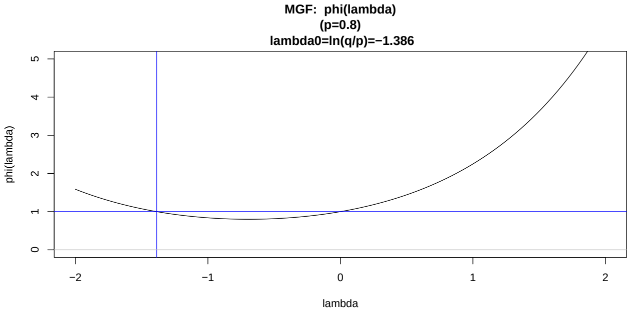
- $S_n = \sum_1^n X_i, \quad S_0 = 0$
- Hit Levels: $+A$, and $-B$
- $\tau = min\{n : S_n = +A, \ \ or \ \ S_n = -B\}$

Moment Generating Function of X_n:

$$\phi(\lambda) = E\left(e^{\lambda X_n}\right) = pe^{\lambda} + qe^{-\lambda}$$

- $Y_n = \frac{e^{\lambda X_n}}{\phi(\lambda)}$ are independent r.v's with $E(Y_n) \equiv 1$.
- $M_n = \Pi_1^n Y_i = e^{\lambda S_n}/[\phi(\lambda)]^n$ is a martingale by Example 4
- Solving $\phi(\lambda) = 1$ for λ :

Random Walk with Bias



Note: $e^{\lambda} = q/p$ solves $\phi(\lambda) = 1$

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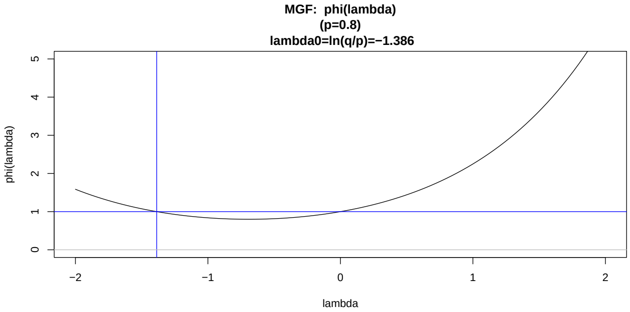

Note: $e^{\lambda} = q/p$ solves $\phi(\lambda) = 1$

So: $M_n = e^{\lambda S_n} = (q/p)^{S_n}$ is a martingale

Solve for $P(S_\tau = k)$

- $M_n = (q/p)^{S_n}$ is a martingale
- $M_{n\wedge\tau}$ is a martingale by the Stopping Time Theorem
- $E(M_{n\wedge\tau}) = 1$, for all n
- $E(M_\tau) = \lim_{n\to\infty} E(M_{n\wedge\tau}) = 1$

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$$1 = E(M_\tau) \quad = \quad (q/p)^A \cdot [P(S_\tau = +A)) + (q/p)^{-B} \cdot [P(S_\tau = -B)]$$

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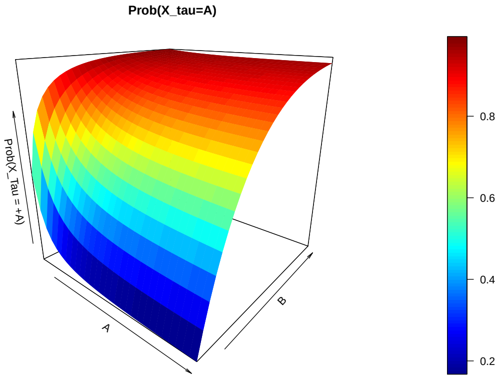
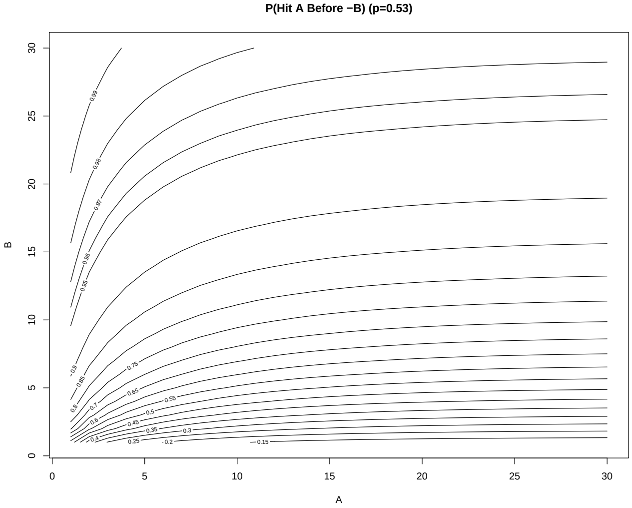
$$\begin{aligned} 1 = E(M_\tau) &= (q/p)^A \cdot [P(S_\tau = +A)) + (q/p)^{-B} \cdot [P(S_\tau = -B)] \\ &= (q/p)^A \cdot [P(S_\tau = +A)] + (q/p)^{-B} \cdot [1 - P(S_\tau = +A)] \end{aligned}$$

$$\implies P(S_\tau = +A) = \frac{(q/p)^B - 1}{(q/p)^{A+B} - 1}.$$

```
> fcn.probA=function(A,B){p=0.53; q=1-p ;
+   probA=((q/p)^B -1 )/((q/p)^(A+B) -1);return(probA)}
> Agrid=seq(1,30) ; Bgrid=seq(1,30)
> vfcn.probA<-Vectorize(fcn.probA)
> zz=outer(Agrid, Bgrid,vfcn.probA)
> persp3D(Agrid,Bgrid,zz,xlab="A",ylab="B",zlab="Prob(X_Tau = +A)",
+     theta=40, phi=20, axes=TRUE,scale=2,box=TRUE,
+     nticks=5,main="Prob(X_tau=A) ")
```



```
> contour(Agrid,Bgrid,zz,xlab="A",ylab="B",zlab="Prob(X_tau=+A)",
+         levels=c(seq(0.05,.95,.05),.96,.97,.98,.99))
> title(main="P(Hit A Before -B) (p=0.53)")
```



Markov Processes

- $\{X_t\}$, a stochastic process

  $t \in \{0, 1, 2, ...\}$ for discrete-time process

  $t \in \{t : t \geq 0\}$ for continuous-time process

- $\{X_t\}$ is a Markov Process if
  - For any times of the process, $u < s < t$
  - Given the value $X_s = x_s$, the process value

X_t is independent of X_u, for all $u < s$.

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X_t is independent of X_u, for all $u < s$.

$$[X_t \mid X_0 = x_0, X_1 = x_1, \ldots, X_s = x_s] \equiv^* [X_t \mid X_s = x_s]$$

($\equiv^*$ means identical probability distributions)

Discrete-time Markov Chain

- State space: $S = \{i : i = 0, 1, 2, \ldots\}$ (finite or countable)
- Time index set: $T = \{n : n = 0, 1, 2, \ldots\}$
- Markov Property:

$$
\begin{aligned}
& Pr\{X_{n+1} = j \quad | \quad X_0 = i_0, \ldots, X_{n-1} = i_{n-1}, X_n = i\} \\
= \;& Pr\{X_{n+1} = j \quad | \quad X_n = i\}
\end{aligned}
$$

for all states $i_0, \ldots, i_{n-1}, i, j$ and all times n

One-Step Transition Probability

$$P_{i,j}^{n,n+1} = Pr(X_{n+1} = j \mid X_n = i), \quad i, j, \in S, \ n \in T$$

Together with $p_i = Pr(x_0 = i), i \in S$ completely specifies stochastic process distribution.

Stationary Markov Process

Stationary Transition Probabilities

$$P_{i,j}^{n,n+1} = P_{i,j} \ \ (\textit{no dependence on } n)$$

Stationary Markov Process

Stationary Transition Probabilities

$$P_{i,j}^{n,n+1} = P_{i,j} \ \ (\textit{no dependence on } n)$$

Stationary Transition Probability Matrix

$$P = ||P_{i,j}||$$

Properties:

- $P_{i,j} \geq 0$ for all i, j
- $\sum_j P_{i,j} = 1$ for every i

The complete probability distribution of $\{X_n, n = 0, 1, \ldots\}$ is specified by:

- Stationary transition probability matrix
  $$P = ||P_{i,j}||$$
- Initial probabilities:
  $$p_i = Pr\{X_0 = i\}, i = 0, 1, \ldots$$

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- Stationary transition probability matrix
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- Initial probabilities:
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To compute any probabilities of the process. It is sufficient to compute

$$P(X_0 = i_0, X_1 = i_1, \ldots, X_n = i_n)$$

for all times n and all states $i_0, i_1, \ldots, i_n$.

$$P(X_0 = i_0, X_1 = i_1, \ldots, X_n = i_n)$$

$$= P(X_0 = i_0, X_1 = i_1, \ldots, X_{n-1} = i_{n-1}) \times P(X_n = i_n \mid X_0 = i_0, X_1 = i_1, \ldots, X_{n-1} = i_{n-1})$$

$$P(X_0 = i_0, X_1 = i_1, \ldots, X_n = i_n)$$

$$\begin{aligned} &= P(X_0 = i_0, X_1 = i_1, \ldots, X_{n-1} = i_{n-1}) \times \\ &\qquad P(X_n = i_n \mid X_0 = i_0, X_1 = i_1, \ldots, X_{n-1} = i_{n-1}) \\ &= P(X_0 = i_0, X_1 = i_1, \ldots, X_{n-1} = i_{n-1}) \times \\ &\qquad P(X_n = i_n \mid X_{n-1} = i_{n-1}) \end{aligned}$$

$$
\begin{aligned}
&P(X_0 = i_0, X_1 = i_1, \ldots, X_n = i_n) \\
&= P(X_0 = i_0, X_1 = i_1, \ldots, X_{n-1} = i_{n-1}) \times \\
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&= P(X_0 = i_0, X_1 = i_1, \ldots, X_{n-1} = i_{n-1}) \times \\
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&= P(X_0 = i_0, X_1 = i_1, \ldots, X_{n-1} = i_{n-1}) \times P_{i_{n-1}, i_n}
\end{aligned}
$$

$$
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Induction on n gives:

$$
\begin{aligned}
&P(X_0 = i_0, X_1 = i_1, \ldots, X_n = i_n) \\
&\qquad = p_{i_0} P_{i_0, i_1} \cdot P_{i_1, i_2} \cdots \times P_{i_{n-2}, i_{n-1}} P_{i_{n-1}, i_n}
\end{aligned}
$$

$$P(X_0 = i_0, X_1 = i_1, \ldots, X_n = i_n)$$

$$\begin{aligned}
&= P(X_0 = i_0, X_1 = i_1, \ldots, X_{n-1} = i_{n-1}) \times \\
&\qquad P(X_n = i_n \mid X_0 = i_0, X_1 = i_1, \ldots, X_{n-1} = i_{n-1}) \\
&= P(X_0 = i_0, X_1 = i_1, \ldots, X_{n-1} = i_{n-1}) \times \\
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Induction on n gives:

$$P(X_0 = i_0, X_1 = i_1, \ldots, X_n = i_n) = p_{i_0} P_{i_0, i_1} \cdot P_{i_1, i_2} \cdots \times P_{i_{n-2}, i_{n-1}} P_{i_{n-1}, i_n}$$

Note use of

- Joint prob = (marginal prob) $\times$ (conditional prob)
- Markov property
- Definition of $P_{i,j}$

Multi-Step Transition Probabilities

Matrix of $n-$step transition probabilities

$$P^{(n)} = ||P_{i,j}^{(n)}||$$

Where:

- $P_{i,j}^{(n)} = Pr\{X_{m+n} = j \mid X_m = i\}$
  for all states $i, j \in S$
  for all increments $n \ (= 1, 2, \ldots)$ after a fixed time $m \in T$
- Stationary case assumed: No Dependence on time m

Theorem: $n-$step transition probabilities of a Markov chain satisfy:

- $P_{i,j}^{(n)} = \sum_{k=0}^{\infty} P_{i,k} P_{k,j}^{(n-1)}$
  with $P_{i,j}^{(0)} = \delta_{i,j}$ ($= 1$ if $i = j$, and $= 0$ if $i \neq j$)
- $P^{(n)} = P \times P \times \cdots P = P^n$

Proof:

- Apply First-Step Analysis analysis
- Apply Markov property
- Apply Stationarity
- Induce result.

$$P_{i,j}^{(n)} = Pr\{X_n = j \mid X_0 = i\} = \sum_{k=0}^{\infty} Pr\{X_n = j, X_1 = k \mid X_0 = i\}$$

$$
\begin{aligned}
P_{i,j}^{(n)} &= Pr\{X_n = j \mid X_0 = i\} = \sum_{k=0}^{\infty} Pr\{X_n = j, X_1 = k \mid X_0 = i\} \\
&= \sum_{k=0}^{\infty} Pr\{X_1 = k \mid X_0 = i\} \times Pr\{X_n = j \mid X_1 = k, X_0 = i\}
\end{aligned}
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$$
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&= \sum_{k=0}^{\infty} P_{i,k} \times Pr\{X_{n-1} = j \mid X_0 = k\}
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&= \sum_{k=0}^{\infty} P_{i,k} \times P_{k,j}^{(n-1)}
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\end{aligned}
$$

Matrix Equation:

$$
P^{(n)} = P \times P^{(n-1)}
$$

Marginal Probability Distribution of X_n

The marginal distribution of X_n for a fixed time n is computed using:

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  $$p_j = Pr\{X_0 = j\} \;\; j = 0, 1, \ldots$$

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- Initial probabilities:
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- $n-$step probabilities given by transition matrix $P^{(n)}$

For every outcome state k of X_n:
$$p_k^{(n)} = Pr\{X_n = k\} = \sum_{j=0}^{\infty} p_j P_{j,k}^{(n)}$$

Markov Chain Examples

Credit Ratings

Standard and Poors Ratings		
Investment Grade	AAA	Highest Grade
	AA	High Grade
	A	Upper Medium Grade
	BBB	Medium Grade
Speculative Grade	BB	Lower Medium Grade
	B	Speculative
	CCC	Poor Standing
	CC	Highly Speculative
	C	Lowest Quality, No Interest
	D	Default

Rating agencies (e.g. Standard and Poors) continuously update credit ratings on:

- Corporate bonds
- Sovereign Foreign Currencies

Migration of Credit Ratings

Corporate Credit Ratings Migration:

Initial Rating	Rating at year end (%)							
	AAA	**AA**	**A**	**BBB**	**BB**	**B**	**CCC**	**Default**
AAA	43.78	53.42	1.65	0.71	0.29	0.11	0.02	0.01
AA	0.60	90.60	6.20	1.45	0.93	0.16	0.04	0.01
A	0.22	2.84	92.97	3.12	0.56	0.14	0.07	0.07
BBB	2.67	3.29	12.77	75.30	5.07	0.60	0.14	0.17
BB	0.19	3.58	8.28	9.97	55.20	17.17	4.53	1.08
B	0.12	0.50	20.69	1.05	0.25	55.40	17.05	4.95
CCC	0.04	0.11	6.28	0.30	0.12	41.53	32.46	19.15

(See Table 6. CreditMetrics Technical Document, 2007 RiskMetrics Group, p.88)

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S&P Global Market Intelligence, 2019 Annual Sovereign Default Study and Ratings Transitions

`https://www.spglobal.com/ratings/en/research/articles/200429-default-transition-and-recovery-2019-annual-global-corporate-default-and-rating-transition-study-`

Markov Chain Examples

Stock Price Dynamics of AAPL

- Daily Stock Prices: $P_t, t = 1, 2, \ldots$
- Up days: $P_t > P_{t-1}$
- Down Days: $P_t < P_{t-1}$
- Define state2DAY on day t: States

state2Day	UU	Up day t-1
Up day t		
UD	Up day t-1	Down day t
DU	Down day t-1	Up day t
DD	Down day t-1	Down day t

- Markov Chain model for state2DAY

R Package for Markov Chains

- Spedicato (2017) Discrete Time Markov Chains with R

  https://journal.r-project.org/archive/2017/RJ-2017-036/index.html
- R Script/pdf: MC_Example1.r

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Fall 2024

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