Time Series Analysis

MIT 18.642

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Time Series: Introduction

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The stochastic behavior of $\{X_t\}$ is determined by specifying the probability density/mass functions (pdf's)

$$p(x_{t_1}, x_{t_2}, \ldots, x_{t_m})$$

for all finite collections of time indexes

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Definition: A time series $\{X_t\}$ is **Strictly Stationary** if

$$p(t_1 + \tau, t_2 + \tau, \ldots, t_m + \tau) = p(t_1, t_2, \ldots, t_m),$$
$$\forall \tau,\ \forall m,\ \forall (t_1, t_2, \ldots, t_m).$$

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(Invariance under time translation)

Covariance Stationarity

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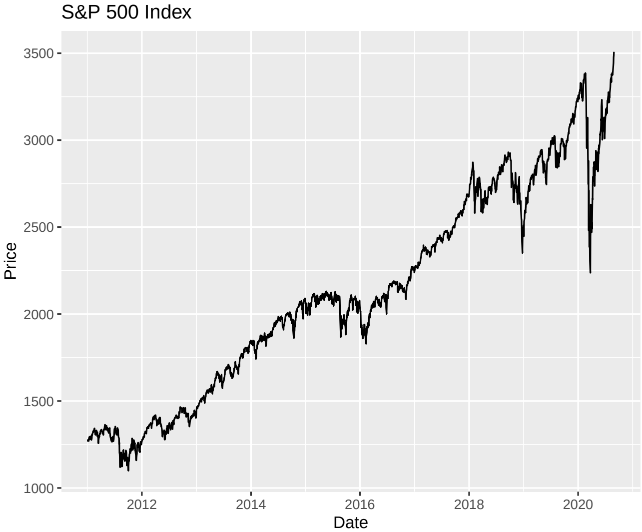
Definition: The **auto-correlation function** of $\{X_t\}$ is

$$\begin{aligned} \rho(\tau) &= \frac{Cov(X_t, X_{t+\tau})}{\sqrt{Var(X_t) \cdot Var(X_{t+\tau})}} \\ &= \frac{\gamma(\tau)}{\gamma(0)} \end{aligned}$$

Financial Time Series



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Exploratory Analysis of Financial Time Series

See: TimeSeries4plots.pdf
TimeSeries4acfplots.pdf

Representation Theorem

Wold Representation Theorem: Any zero-mean covariance stationary time series $\{X_t\}$ can be decomposed as $X_t = V_t + S_t$ where

- $\{V_t\}$ is a linearly deterministic process, i.e., a linear combination of past values of V_t with constant coefficients.
- $S_t = \sum_{i=0}^{\infty} \psi_i \eta_{t-i}$ is a moving average process of error terms, where
  - $\psi_0 = 1$, $\sum_{i=0}^{\infty} \psi_i^2 < \infty$
  - $\{\eta_t\}$ is linearly unpredictable white noise, i.e.,

    $$E(\eta_t) = 0,\ E(\eta_t^2) = \sigma^2,\ E(\eta_t \eta_s) = 0\ \forall t,\ \forall s \neq t,$$

    and $\{\eta_t\}$ is uncorrelated with $\{V_t\}$:

    $$E(\eta_t V_s) = 0,\ \forall t, s$$

Intuitive Application of the Wold Representation Theorem

Suppose we want to specify a covariance stationary time series $\{X_t\}$ to model actual data from a real time series
$$\{x_t, t = 0, 1, \dots, T\}$$
Consider the following strategy:

- Initialize a parameter p, the number of past observations in the linearly deterministic term of the Wold Decomposition of $\{X_t\}$
- Estimate the linear projection of X_t on $(X_{t-1}, X_{t-2}, \dots, X_{t-p})$
  - Consider an estimation sample of size n with endpoint $t_0 \leq T$.
  - Let $\{j = -(p-1), \dots, 0, 1, 2, \dots n\}$ index the subseries of $\{t = 0, 1, \dots, T\}$ corresponding to the estimation sample and define $\{y_j : y_j = x_{t_0-n+j}\}$, (with $t_0 \geq n + p$)
  - Define the vector $\mathbf{Y}_{(n \times 1)}$ and matrix $\mathbf{Z}_{(n \times [p+1])}$ as:

- Estimate the linear projection of X_t on $(X_{t-1}, X_{t-2}, \ldots, X_{t-p})$ (continued)

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \quad \mathbf{Z} = \begin{bmatrix} 1 & y_0 & y_{-1} & \cdots & y_{-(p-1)} \\ 1 & y_1 & y_0 & \cdots & y_{-(p-2)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & y_{n-1} & y_{n-2} & \cdots & y_{n-p} \end{bmatrix}$$

  - Apply OLS to specify the projection:

$$\begin{aligned} \hat{\mathbf{y}} &= \mathbf{Z}(\mathbf{Z}^T\mathbf{Z})^{-1}\mathbf{Z}\mathbf{y} \\ &= \hat{P}(Y_t \mid Y_{t-1}, Y_{t-2}, \ldots Y_{t-p}) \\ &= \hat{\mathbf{y}}^{(p)} \end{aligned}$$

- Compute the projection residual

$$\hat{\boldsymbol{\epsilon}}^{(p)} = \mathbf{y} - \hat{\mathbf{y}}^{(p)}$$

- Apply time series methods to the time series of residuals $\{\hat{\epsilon}_j^{(p)}\}$ to specify a moving average model:

  $$\epsilon_t^{(p)} = \sum_{i=0}^{\infty} \psi_j \eta_{t-i}$$

  yielding $\{\hat{\psi}_j\}$ and $\{\hat{\eta}_t\}$, estimates of parameters and innovations.
- Conduct a case analysis diagnosing consistency with model assumptions
  - Evaluate orthogonality of $\hat{\epsilon}^{(p)}$ to Y_{t-s}, $s > p$.
    If evidence of correlation, increase p and start again.
  - Evaluate the consistency of $\{\hat{\eta}_t\}$ with the white noise assumptions of the theorem.
    If evidence otherwise, consider revisions to the overall model
    - Changing the specification of the moving average model.
    - Adding additional 'deterministic' variables to the projection model.

Note:

- Theoretically,

$$\lim_{p\to\infty} \hat{\mathbf{y}}^{(p)} = \hat{\mathbf{y}} = P(Y_t \mid Y_{t-1}, Y_{t-2}, \ldots)$$

  but if $p \to \infty$ is required, then $n \to \infty$ while $p/n \to 0$.
- Useful models of covariance stationary time series have
  - Modest finite values of p and/or include
  - Moving average models depending on a parsimonious number of parameters.

Lag Operator $L()$

Definition The lag operator $L()$ shifts a time series back by one time increment. For a time series $\{X_t\}$:

$$L(X_t) = X_{t-1}.$$

Applying the operator recursively we define:

$$\begin{aligned} L^0(X_t) &= X_t \\ L^1(X_t) &= X_{t-1} \\ L^2(X_t) &= L(L(X_t)) = X_{t-2} \\ &\cdots \\ L^n(X_t) &= L(L^{n-1}(X_t)) = X_{t-n} \end{aligned}$$

Inverses of these operators are well defined as:

$$L^{-n}(X_t) = X_{t+n}, \text{ for } n = 1, 2, \ldots$$

Wold Representation with Lag Operators

The Wold Representation for a covariance stationary time series $\{X_t\}$ can be expressed as

$$\begin{aligned} X_t &= \sum_{i=0}^{\infty} \psi_i \eta_{t-i} + V_t \\ &= \sum_{i=0}^{\infty} \psi_i L^i(\eta_t) + V_t \\ &= \psi(L)\eta_t + V_t \end{aligned}$$

where $\psi(L) = \sum_{i=0}^{\infty} \psi_i L^i$.

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$$IR(j) = \frac{\partial X_t}{\partial \eta_{t-j}} = \psi_j.$$

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Definition The **Impulse Response Function** of the covariance stationary process $\{X_t\}$ is

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The **long-run cumulative response** of $\{X_t\}$ is

$$\sum_{i=0}^{\infty} IR(j) = \sum_{i=0}^{\infty} \psi_i = \psi(L) \text{ with } L = 1.$$

Equivalent Auto-regressive Representation

Suppose that the operator $\psi(L)$ is invertible, i.e.,

$\psi^{-1}(L) = \sum_{i=0}^{\infty} \psi_i^* L^i$ such that

$\psi^{-1}(L)\psi(L) = I = L^0$.

Then, assuming $V_t = 0$ (i.e., X_t has been adjusted to $X_t^* = X_t - V_t$), we have the following equivalent expressions of the time series model for $\{X_t\}$

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\begin{aligned}
X_t &= \psi(L)\eta_t \\
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Definition When $\psi^{-1}(L)$ exists, the time series $\{X_t\}$ is **Invertible** and has an auto-regressive representation:

$$X_t = (\sum_{i=0}^{\infty} \psi_i^* X_{t-i}) + \eta_t$$

Definition: The times series $\{X_t\}$ follows the $ARMA(p,q)$ **Model** with auto-regressive order p and moving-average order q if

$$\begin{aligned} X_t &= \mu + \phi_1(X_{t-1} - \mu) + \phi_2(X_{t-1} - \mu) + \cdots \phi_p(X_{t-p} - \mu) \\ &\quad + \eta_t + \theta_1\eta_{t-1} + \theta_2\eta_{t-2} + \cdots \theta_q\eta_{t-q} \end{aligned}$$

where $\{\eta_t\}$ is $WN(0, \sigma^2)$, "**White Noise**" with

$$\begin{aligned} E(\eta_t) &= 0, \quad \forall t \\ E(\eta_t^2) &= \sigma^2 < \infty, \quad \forall t \text{ , and } E(\eta_t \eta_s) = 0, \ \forall t \neq s \end{aligned}$$

ARMA(p,q) Models

Definition: The times series $\{X_t\}$ follows the $ARMA(p,q)$ **Model** with auto-regressive order p and moving-average order q if

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With lag operators

$$\phi(L) = (1 - \phi_1 L - \phi_2 L^2 - \cdots \phi_p L^P) \text{ and}$$
$$\theta(L) = (1 + \theta_1 L + \theta_2 L^2 + \cdots + \theta_q L^q)$$

we can write

$$\phi(L) \cdot (X_t - \mu) = \theta(L)\eta_t$$

and the Wold decomposition is

$$X_t = \mu + \psi(L)\eta_t, \text{ where } \psi(L) = [\phi(L)])^{-1}\theta(L)$$

AR(p) Models

Order-p Auto-Regression Model: AR(p)

$$\phi(L) \cdot (X_t - \mu) = \eta_t \text{ where}$$
$$\{\eta_t\} \text{ is } WN(0, \sigma^2) \text{ and}$$
$$\phi(L) = 1 - \phi_1 L - \phi_2 L^2 - \cdots + \phi_p L^p$$

Properties:

- Linear combination of $\{X_t, X_{t-1}, \ldots X_{t-p}\}$ is $WN(0, \sigma^2)$.
- X_t follows a linear regression model on explanatory variables $(X_{t-1}, X_{t-2}, \ldots, X_{t-p})$, i.e

$$X_t = c + \sum_{j=1}^{p} \phi_j X_{t-j} + \eta_t$$

where $c = \mu \cdot \phi(1)$, (replacing L by 1 in $\phi(L)$).

Stationarity Conditions

Consider $\phi(z)$ replacing L with a complex variable z.

$$\phi(z) = 1 - \phi_1 z - \phi_2 z^2 - \cdots \phi_p z^p.$$

Let $\lambda_1, \lambda_2, \ldots \lambda_p$ be the p roots of $\phi(z) = 0$.

$$\phi(L) = (1 - \frac{1}{\lambda_1} L) \cdot (1 - \frac{1}{\lambda_2} L) \cdots (1 - \frac{1}{\lambda_p} L)$$

Claim: $\{X_t\}$ is covariance stationary if and only if all the roots of $\phi(z) = 0$ (the"**characteristic equation**") lie outside the unit circle $\{z : |z| \leq 1\}$, i.e., $|\lambda_j| > 1$, $j = 1, 2, \ldots, p$

- For complex number λ: $|\lambda| > 1$,

$$\begin{aligned} (1 - \frac{1}{\lambda} L)^{-1} &= 1 + (\frac{1}{\lambda}) L + (\frac{1}{\lambda})^2 L^2 + (\frac{1}{\lambda})^3 L^3 + \cdots \\ &= \sum_{i=0}^{\infty} (\frac{1}{\lambda})^i L^i \end{aligned}$$

- $\phi^{-1}(L) = \prod_{j=1}^{p} \left[\left(1 - \frac{1}{\lambda_j} L\right)^{-1} \right]$

AR(1) Model

Suppose $\{X_t\}$ follows the $AR(1)$ process, i.e.,

$$X_t - \mu = \phi(X_{t-1} - \mu) + \eta_t, \; t = 1, 2, \ldots$$

where $\eta_t \sim WN(0, \sigma^2)$.

- The characteristic equation for the $AR(1)$ model is
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  $$E(X_t) = \mu$$

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- For $\phi : |\phi| < 1$, the Wold decomposition of the $AR(1)$ model is: $X_t = \mu + \sum_{j=0}^{\infty} \phi^j \eta_{t-j}$
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  - For $\phi : 0 > \phi > -1$, the $AR(1)$ process exhibits oscillating exponential mean-reversion to μ

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  - For $\phi : 0 < \phi < 1$, the $AR(1)$ process exhibits exponential mean-reversion to μ
  - For $\phi : 0 > \phi > -1$, the $AR(1)$ process exhibits oscillating exponential mean-reversion to μ
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Examples of $AR(1)$ Models (mean reverting with $0 < \phi < 1$)

- Interest rates (Ornstein Uhlenbeck Process; Vasicek Model)
- Interest rate spreads
- Real exchange rates
- Valuation ratios (dividend-to-price, earnings-to-price)

Yule Walker Equations for AR(p) Processes

Second Order Moments of $AR(p)$ Processes

From the specification of the $AR(p)$ model:

$$(X_t - \mu) = \phi_1(X_{t-1} - \mu) + \phi_2(X_{t-1} - \mu) + \cdots + \phi_p(X_{t-p} - \mu) + \eta_t$$

we can write the **Yule-Walker Equations** ($j = 0, 1, \ldots$)

$$\begin{aligned} E[(X_t - \mu)(X_{t-j} - \mu)] &= \phi_1 E[(X_{t-1} - \mu)(X_{t-j} - \mu)] \\ &\quad + \phi_2 E[(X_{t-1} - \mu)(X_{t-j} - \mu)] + \\ &\quad \cdots + \phi_p E[(X_{t-p} - \mu)(X_{t-j} - \mu)] \\ &\quad + E[\eta_t(X_{t-j} - \mu)] \\ \gamma(j) &= \phi_1\gamma(j-1) + \phi_2\gamma(j-2) + \\ &\quad \cdots + \phi_p\gamma(j-p) + \delta_{0,j}\sigma^2 \end{aligned}$$

Equations $j = 1, 2, \ldots p$ yield a system of p linear equations in ϕ_j:

Yule-Walker Equations

$$\begin{pmatrix} \gamma(1) \\ \gamma(2) \\ \vdots \\ \gamma(p) \end{pmatrix} = \begin{bmatrix} \gamma(0) & \gamma(-1) & \gamma(-2) & \cdots & \gamma(-(p-1)) \\ \gamma(1) & \gamma(0) & \gamma(-1) & \cdots & \gamma(-(p-2)) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \gamma(p-1) & \gamma(p-2) & \gamma(p-3) & \cdots & \gamma(0)) \end{bmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_p \end{pmatrix}$$

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  $$\hat{\sigma}^2 = \hat{\gamma}(0) - \sum_{j-1}^{p} \hat{\phi}_j\hat{\gamma}(j)$$
- When all the estimates $\hat{\gamma}(j)$ and $\hat{\mu}$ are unbiased, then the Yule-Walker estimates apply the **Method of Moments** Principle of Estimation.

MA(q) Models

Order-q Moving-Average Model: MA(q)

$$(X_t - \mu) = \theta(L)\eta_t, \text{ where}$$
$$\{\eta_t\} \text{ is } WN(0, \sigma^2) \text{ and}$$
$$\theta(L) = 1 + \theta_1 L + \theta_2 L^2 + \cdots + \theta_q L^q$$

Properties:

- The process $\{X_t\}$ is invertible if all the roots of $\theta(z) = 0$ are outside the complex unit circle.

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$$Cov(X_t, X_{t+j}) = \begin{cases} 0, & j > q \\ \sigma^2 \cdot (\theta_j + \theta_{j+1}\theta_1 + \theta_{j+2}\theta_2 + \cdots \theta_q \theta_{q-j}), & 1 < j \leq q \end{cases}$$

Accommodating Non-Stationarity by Differencing

Many economic time series exhibit non-stationary behavior consistent with random walks. Box and Jenkins advocate removal of non-stationary trending behavior using

Differencing Operators:

$$\begin{aligned}\Delta &= 1 - L \\ \Delta^2 &= (1-L)^2 = 1 - 2L + L^2 \\ \Delta^k &= (1-L)^k = \sum_{j=0}^{k} \binom{k}{j} (-L)^j, \ \text{(integral } k > 0\text{)}\end{aligned}$$

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Examples of Non-Stationary Processes

Linear Trend Reversion Model: Suppose the model for the time series $\{X_t\}$ is:

$$X_t = TD_t + \eta_t, \text{ where}$$

- $TD_t = a + bt$, a deterministic (linear) trend
- $\eta_t \sim AR(1)$, i.e.,

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The differenced process $\{\Delta X_t\}$ can be expressed as

$$\begin{aligned} \Delta X_t &= b + \Delta\eta_t \\ &= b + (\eta_t - \eta_{t-1}) \\ &= b + (1 - L)\eta_t \\ &= b + (1 - L)(1 - \phi L)^{-1}\xi_t \end{aligned}$$

Non-Stationary Trend Processes

Pure Integrated Process I(1) for $\{X_t\}$:

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Equivalently:

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Given X_0, we can write $X_t = X_0 + TS_t$ where

$$TS_t = \sum_{j=0}^{t} \eta_j$$

The process $\{TS_t\}$ is a **Stochastic Trend** process with

$$TS_t = TS_{t-1} + \eta_t, \text{ where } \{\eta_t\} \text{ is } WN(0, \sigma^2).$$

Note:

- The Stochastic Trend process is not perfectliy predictable.
- The process $\{X_t\}$ is a **Simple Random Walk** with white-noise steps. It is non-stationary because given X_0:
  - $Var(X_t) = t\sigma^2$
  - $Cov(X_t, X_{t-j}) = (t - j)\sigma^2$ for $0 < j < t$.
  - $Corr = (X_t, X_{t-j}) = \sqrt{t - j}/\sqrt{t} = \sqrt{1 - j/t}$

ARIMA(p,d,q) Models

Definition: The time series $\{X_t\}$ follows an $ARIMA(p, d, q)$ model ("Integrated ARMA") if $\{\Delta^d X_t\}$ is stationary (and non-stationary for lower-order differencing) and follows an $ARMA(p, q)$ model.

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Issues:

- Determining the order of differencing required to remove time trends (deterministic or stochastic).
- Estimating the unknown parameters of an $ARIMA(p, d, q)$ model.
- Model Selection: choosing among alternative models with different (p, d, q) specifications.

Estimation of ARMA Models

Maximum-Likelihood Estimation

- Assume that $\{\eta_t\}$ are i.i.d. $N(0, \sigma^2)$ r.v.'s.
- Express the $ARMA(p, q)$ model in state-space form.
- Apply the prediction-error decomposition of the log-likelihood function.

Limited Information Maximum-Likelihood (LIML) Method

- Condition on the first p values of $\{X_t\}$
- Assume that the first q values of $\{\eta_t\}$ are zero.

Full Information Maximum-Likelihood (FIML) Method

- Use the stationary distribution of the first p values to specify the exact likelihood.

Model Selection

Statistical model selection critera are used to select the orders (p, q) of an ARMA process:

- Fit all $ARMA(p, q)$ models with $0 \leq p \leq p_{max}$ and $0 \leq q \leq q_{max}$, for chosen values of maximal orders.
- Let $\tilde{\sigma}^2(p, q)$ be the MLE of $\sigma^2 = Var(\eta_t)$, the variance of ARMA innovations under Gaussian/Normal assumption.
- Choose (p, q) to minimize one of:

  Akaike Information Criterion

  $$AIC(p, q) = log(\tilde{\sigma}^2(p, q)) + 2\frac{p+q}{n}$$

  Bayes Information Criterion

  $$BIC(p, q) = log(\tilde{\sigma}^2(p, q)) + log(n)\frac{p+q}{n}$$

  Hannan-Quinn Criterion

  $$HQ(p, q) = log(\tilde{\sigma}^2(p, q)) + 2log(log(n))\frac{p+q}{n}$$

Testing for Stationarity/Non-Stationarity

Dickey-Fuller (DF) Test : Suppose $\{X_t\}$ follows the $AR(1)$ model

$$X_t = \phi X_{t-1} + \eta_t, \text{ with } \{\eta_t\} \text{ a } WN(0, \sigma^2).$$

Consider testing the following hypotheses:

H_0: $\phi = 1$ (unit root, non-stationarity)

H_1: $|\phi| < 1$ (stationarity)

("Autoregressive Unit Root Test")

- Fit the $AR(1)$ model by least squares and define the test statistic: $t_{\phi=1} = \frac{\hat{\phi}-1}{se(\hat{\phi})}$

  where $\hat{\phi}$ is the least-squares estimate of ϕ and $se(\hat{\phi})$ is the least-squares estimate of the standard error of $\hat{\phi}$.

  - Under H_1: if $|\phi| < 1$, then $\sqrt{T}(\hat{\phi} - \phi) \xrightarrow{d} N(0, (1 - \phi^2))$.

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  - Under H_0: if $\phi = 1$, then $\hat{\phi}$ is super-consistent with rate $(1/T)$,

    $T \cdot t_{\phi=1}$ has DF distribution.

References on Tests for Stationarity/Non-Stationarity*

Unit Root Tests (H_0 : Nonstationarity)

- Dickey and Fuller (1979): Dickey-Fuller (DF) Test
- Said and Dickey (1984): Augmented Dickey-Fuller (ADF) Test
- Phillips and Perron (1988) Unit root (PP) tests
- Elliot, Rothenberg, and Stock (2001) Efficient unit root (ERS) test statistics.

Stationarity Tests (H_0 : stationarity)

- Kwiatkowski, Phillips, Schmidt, and Shin (1922): KPSS test.

* Optional reading