Stochastic Calculus

MIT 18.642

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Brownian Motion with Drift

Brownian Motion with Drift

- Let $\{B(t), t \geq 0; B(0) = 0\}$ be a Standard Brownian Motion
- Define $\{X(t); t \geq 0; X(0) = 0\}$
  $$X(t) = \mu t + \sigma B(t), \text{ for } t \geq 0.$$
  $\mu =$ **drift parameter**
  $\sigma^2 =$ **variance parameter**

Key Properties of Brownian Motion with Drift

- Independent Increments
- $Var[X(t) - X(s)] \propto |t - s|$
  (Same as for Standard Brownian Motion $\mu = 0$, $\sigma = 1$.)

Brownian Motion with Drift

Infinitesimal, One-Step Analysis:

- Conditional Distribution of $X(t+\Delta t)$ given $X(t) = x$

$$\begin{aligned} X(t+\Delta t) &= \mu(t+\Delta t) + \sigma B(t+\Delta t) \\ &= [\mu t + \sigma B(t)] + \mu\Delta t + \sigma[B(t+\Delta t) - B(\Delta t)] \\ &= X(t) + \mu\Delta t + \sigma\Delta B(t) \end{aligned}$$

- Increment of $X(\cdot)$ in terms of increments Δt and $\Delta B(t)$

$$\Delta X = X(t+\Delta t) - X(t) = \mu\Delta t + \sigma\Delta B(t)$$

Properties:

- $E[\Delta X] = \mu\Delta t$
- $Var[\Delta X] = \sigma^2\Delta t$
- Exact distribution: $\Delta X \sim N(\mu\Delta t, \sigma^2\Delta t)$.
- As $\Delta t \to 0$

$$\begin{aligned} E[(\Delta X)^2 \mid X(t) = x] &= \sigma^2\Delta t + (\mu\Delta t)^2 \\ &= \sigma^2\Delta t + o(\Delta t). \\ E[(\Delta X)^c \mid X(t) = x] &= o(\Delta t) \quad \text{for } c > 2 \end{aligned}$$

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Note: As $\Delta t \searrow 0$, ignore terms of $o(\Delta t)$.

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Ito Processes

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- Extend drift: $\mu = \mu(X_t, t)$
- Extend volatility: $\sigma = \sigma(X_t, t)$

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dB_t$$

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(Requires definition of Ito Integrals)

Consider the Brownian motion process $\{B_t, t \geq 0\}$ with:

- Probability model $(\Omega, \{\mathcal{F}_t\}, P)$
- $\Omega = \{\omega\}$ set of all paths ω
- $\{\mathcal{F}_t, t \geq 0\}$: filtration of the process
  Events/sets in $\mathcal{F}_t$ (subsets of Ω) that are
  measurable with respect to
  $\{B_s, s \leq t\}$ (information set up to time t)
- $P(\cdot)$: probability measure on Ω
  For all $A \in \mathcal{F}_t$, $P(A)$ well defined.
- $B_t = B_t(\omega)$: a specific path

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- Define for each path ω: $\mathrm{I}(f)(\omega) = \int_0^T f(B_s(\omega))dB_s(\omega)$.
- $\mathrm{I}(f)(\omega)$ a random variable with $E[\mathrm{I}(f)]$ and $Var[\mathrm{I}(f)]$

Computing $\mathrm{I}(f) = \int_0^T f(B_s)dB_s$ for Special Cases of f

Case 1: $f(x) \equiv 1$

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Computing Ito Integrals

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$$\begin{aligned} Var[\mathrm{I}(f)(\omega)] &= \sum_{i=0}^{n-1} E[B_{t_i}^2] E([B_{t_{i+1}} - B_{t_i}]^2) \\ &= \sum_{i=0}^{n-1} [t_i][t_{i+1} - t_i] \approx \int_0^T s ds = \frac{1}{2} T^2 \end{aligned}$$

Computing Ito Integrals

Case 4: $f(\omega, s) = \lim_{n\to\infty} f_n(\omega, s) = \lim_{n\to\infty} \sum_{i=0}^{n-1} a_i(\omega) 1(t_i < s \le t_{i+1})$

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Claim: $\mathrm{I}(f)(\omega) = \frac{1}{2} B_T^2(\omega) - \frac{1}{2} T$ (**Note(!!)** for all ω)

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$$\begin{aligned} \sum_{i=0}^{n-1} B_{t_i}(\omega)[B_{t_{i+1}}(\omega) - B_{t_i}(\omega)] &= \sum_{i=0}^{n-1} \left(\frac{1}{2}[B_{t_{i+1}}^2 - B_{t_i}^2]\right) \\ &\quad - \sum_{i=0}^{n-1} \left(\frac{1}{2}[B_{t_{i+1}} - B_{t_i}]^2\right) \end{aligned}$$

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Claim: $\mathrm{I}(f)(\omega) = \frac{1}{2} B_T^2(\omega) - \frac{1}{2} T$ (**Note(!!)** for all ω)

Proof:

$$B_{t_i}(\omega)[B_{t_{i+1}}(\omega) - B_{t_i}(\omega)] = \frac{1}{2}[B_{t_{i+1}}^2 - B_{t_i}^2] - \frac{1}{2}[B_{t_{i+1}} - B_{t_i}]^2$$

$$\begin{aligned} \sum_{i=0}^{n-1} B_{t_i}(\omega)[B_{t_{i+1}}(\omega) - B_{t_i}(\omega)] &= \sum_{i=0}^{n-1} \left(\frac{1}{2}[B_{t_{i+1}}^2 - B_{t_i}^2]\right) \\ &\quad - \sum_{i=0}^{n-1} \left(\frac{1}{2}[B_{t_{i+1}} - B_{t_i}]^2\right) \\ &= \frac{1}{2} B_{t_n}^2 - \frac{1}{2} \sum_{i=0}^{n-1} \left([B_{t_{i+1}} - B_{t_i}]^2\right) \end{aligned}$$

Computing Ito Integrals

Case 4: $f(\omega, s) = \lim_{n\to\infty} f_n(\omega, s) = \lim_{n\to\infty} \sum_{i=0}^{n-1} a_i(\omega) 1(t_i < s \leq t_{i+1})$

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$$\begin{aligned} \Longrightarrow \mathrm{I}(f)(\omega) &= \frac{1}{2} B_T^2 - \frac{1}{2} \lim_{n\to\infty} \sum_{i=0}^{n-1} (B_{t_{i+1}} - B_{t_i})^2 \\ &= \frac{1}{2} B_T^2 - \frac{1}{2} T \end{aligned}$$

Ito Integral: General Case

$$X_t(\omega) = \int_0^t f(\omega, s) dB_s(\omega),\ 0 \le t \le T$$

Ito Process from Ito Integrals

Ito Integral: General Case

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$f(\omega, s)$: function of ω and time s

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  Squared-norm of $f(\omega, s)$ on $\Omega \times [t, t+k]$

Ito Isometry

Ito Integral of $f(\omega, s)$ w.r.t. $\{B_s, 0 \leq s < T\}$

$$\mathrm{I}(f)(\omega) = \int_0^T f(\omega, s) dB_s(\omega)$$

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  - Formalizes limit/convergence of sequences of Ito integrals

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- $||f||_{f^2(dP \times dt)}$ is (squared) norm on
  space of Ito-Process Integrands
  - Formalizes limit/convergence of sequences of Ito integrals
  - Enables focus on closed linear subspaces of
    $L^2(dP \times dt)$ for $\{f(\omega, s)\}$.

Ito Integral As Limiting Riemann Sum

Theorem (Riemann Representation). For any continuous $f : R \to R$, if we take the partition of $[0, T]$ given by $t_i = iT/n$ for $0 \le i \le n$, then we have

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(B_{t_{i-1}})(B_{t_i} - B_{t_{i-1}}) = \int_0^T f(B_s)dB_s$$

Note: Convergence is in Probability

Proposition (Gaussian Integrals). If $f \in C[0, T]$, (a continuous function of time for $t \in [0, T]$) then the Ito Process defined by

$$X_t = \int_0^t f(s)dB_s, \text{ for } s \in [0, T]$$

has the following properties:

- Mean-zero Gaussian process
- Independent increments
- Covariance function

$$Cov(X_s, X_t) = \int_0^{s \wedge t} f^2(u)du.$$

Gaussian Integrals

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has the following properties:

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- Covariance function

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Proof:

- Take the partition of $[0, T]$ given by $t_i = iT/n$ for $0 \le i \le n$
- Choose $t_i^* : t_{i-1} \le t_i^* \le t_i$ for all i. Then

  $$\lim_{n \to \infty} \sum_{i=1}^{n} f(t_i^*)(B_{t_i} - B_{t_{i-1}}) = \int_0^T f(s)dB_s$$

See Section 7.2 of Steele(1980) for details.

Ito's Formula (One Variable)

Theorem (Ito's Formula - Case 1). If $f : R \to R$ has a continuous second derivative, then

$$f(B_t) = f(0) + \int_0^t f'(B_s) dB_s + \frac{1}{2} \int_0^t f''(B_s) ds.$$

- $\int_0^t f'(B_s) dB_s$: zero-mean adaptive Gaussian Process
- $\frac{1}{2} \int_0^t f''(B_s) ds$: adaptive drift term

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'Proof': Apply Taylor Series of $f(\cdot)$

$$f(x+\Delta x) = f(x) + f'(x)\Delta x + \frac{1}{2}f''(x)(\Delta x)^2 + r$$

$$\begin{aligned} f(B_t) - f(B_0) &= \lim_{n\to\infty} \sum_{i=1}^n [f(B_{t_i}) - f(B_{t_{i-1}})] \\ &= \lim_{n\to\infty} \sum_{i=1}^n [f'(B_{t_{i-1}})(B_{t_i} - B_{t_{i-1}}) \\ &\qquad + \frac{1}{2}f''(B_{t_{i-1}})(B_{t_i} - B_{t_{i-1}})^2 + r_i] \\ &= \int_0^t f'(B_s)dB_s + \frac{1}{2}\int_0^t f''(B_s)ds + 0 \end{aligned}$$

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Ito's Formula - Shorthand:

$$\begin{aligned} df(B_s) &= f'(B_s)dB_s + \frac{1}{2}f''(B_s)ds \\ &= \frac{df}{dx}(B_t)dB_s + \frac{1}{2}\frac{d^2f}{dx^2}(B_s)ds \end{aligned}$$

Ito's Formula: Applications

Application:

Solving for $\int_0^t f(B_s)dB_s$

- Find the anti-derivative of $f(\cdot)$
  $$F(\cdot) \in C^2(R) : F'(\cdot) = f(\cdot)$$
- Use the specific anti-derivative with $F(0) = 0$
- Use Taylor Series for $F(B_t)$:
  $$F(B_t) - F(B_0) = \int_0^t f(B_s)dB_s + \frac{1}{2}\int_0^t f'(B_s)ds$$

$$\implies \quad \int_0^t f(B_s)dB_s = F(B_t) - \frac{1}{2}\int_0^t f'(B_s)ds$$

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- Use Taylor Series for $F(B_t)$:
  $$F(B_t) - F(B_0) = \int_0^t f(B_s)dB_s + \frac{1}{2}\int_0^t f'(B_s)ds$$

$$\Longrightarrow \quad \int_0^t f(B_s)dB_s = F(B_t) - \frac{1}{2}\int_0^t f'(B_s)ds$$

Example 1. $F(x) = x$, $f = F' = 1$ and $f' = F'' = 0$

$$\int_0^t dB_s = B_t$$

Ito's Formula: Applications

Application:

Solving for $\int_0^t f(B_s)dB_s$

- Find the anti-derivative of $f(\cdot)$
  $$F(\cdot) \in C^2(R) : F'(\cdot) = f(\cdot)$$
- Use the specific anti-derivative with $F(0) = 0$
- Use Taylor Series for $F(B_t)$:
  $$F(B_t) - F(B_0) = \int_0^t f(B_s)dB_s + \frac{1}{2}\int_0^t f'(B_s)ds$$

$$\implies \quad \int_0^t f(B_s)dB_s = F(B_t) - \frac{1}{2}\int_0^t f'(B_s)ds$$

Example 1. $F(x) = x$, $f = F' = 1$ and $f' = F'' = 0$

$$\int_0^t dB_s = B_t$$

Example 2. $F(x) = \frac{1}{2}x^2$, $f = F' = x$ and $f' = F'' = 1$

$$\int_0^t B_s dB_s = F(B_t) - \frac{1}{2}\int_0^t ds = \frac{1}{2}B_t^2 - \frac{1}{2}t$$

Ito's Formula: Two Variables

Consider functions $f \in C^{1,2}(R^+ \times R)$

- Functions of time and space variables:
  (s, x), $s \in R^+$, and $x \in R$.
- Taylor Series:

$$f(t+\Delta t, x+\Delta x) = f(t,x) + \Delta t \frac{\partial f}{\partial t}(t,x) + \Delta x \frac{\partial f}{\partial x}(t,x) + + \tfrac{1}{2}(\Delta x)^2 \frac{\partial^2 f}{\partial x^2}(t,x) + r$$

Note: $r = o(\Delta t) = r((\Delta t)^2, (\Delta t \Delta x), (\Delta t)^a (\Delta x)^b, a \geq 2, b \geq 3)$

Theorem (Ito's Formula with Space and Time Variables)

For any function $f \in C^{1,2}(R^{\times} R)$,

$$f(t, B_t) = f(0,0) + \int_0^t \frac{\partial f}{\partial t}(s, B_s) ds + \int_0^t \frac{\partial f}{\partial x}(s, B_s) dB_s + + \tfrac{1}{2} \int_0^t \frac{\partial^2 f}{\partial x^2}(s, B_s) ds.$$

Theorem (Ito's Formula with Space and Time Variables)
For any function $f \in C^{1,2}(R^{\times} R)$,

$$f(t, B_t) - f(0,0) \;=\; \int_0^t \left(\frac{\partial f}{\partial t}(s, B_s) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(s, B_s) \right) ds + \int_0^t \frac{\partial f}{\partial x}(s, B_s) dB_s$$

NOTE(!!): $\{f(t, B_t)\}$ may be a **Martingale** if

$$0 = \left(\frac{\partial f}{\partial t}(s, B_s) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(s, B_s) \right)$$

$$\iff \quad \frac{\partial f}{\partial t} = -\frac{1}{2}\frac{\partial^2 f}{\partial x^2} \qquad \text{(PDE Condition for } f\text{)}$$

Additional Condition:

$$E[|f(T, B_T) - f(0,0)|^2] = E[\int_0^T \left(\frac{\partial f(t, B_t)}{\partial x} \right)^2 dt] < \infty.$$

Ito's Formula: Applications

Example 4. $f(t,x) = e^{\sigma x - \sigma^2 t/2}$

$$\frac{\partial f}{\partial t} = -\frac{1}{2}\sigma^2 f, \quad \frac{\partial f}{\partial x} = \sigma f, \quad \text{and } \frac{\partial^2 f}{\partial x^2} = \sigma^2 f$$

Applying Ito's Formula

$$\begin{aligned} f(t,B_t) &= f(0,0) + \int_0^t \left(\frac{\partial f}{\partial t}(s,B_s) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(s,B_s) \right) ds \\ &\qquad + \int_0^t \frac{\partial f}{\partial x}(s,B_s) dB_s \\ &= f(0,0) + \int_0^t 0 ds + \int_0^t \sigma e^{\sigma B_s - \sigma^2 s/2} dB_s \\ &= f(0,0) + \int_0^t \sigma e^{\sigma B_s - \sigma^2 s/2} dB_s \end{aligned}$$

$f(t,B_t)$ is a Martingale:

- $E[f(t,B_t) \mid f(0,0)] = f(0,0) = 1 \qquad$ (for all $t > 0$)

Note: $f(t,B_t) = e^{\sigma B_t - \sigma^2 t/2}$ is a scaled log-normal r.v.

$E[e^{(\sigma B_t)}] = e^{+\frac{1}{2}\sigma^2 t}$, so $E[f(t,B_t)] = 1$ for all t.

Ito's Formulation: Applications

Example 5. Brownian Motion with Drift: The Ruin Problem

- $\{B_t, t \geq 0\}$ standard Brownian motion.
- $X_t = \mu t + \sigma B_t$: Brownian motion with
  drift μ and instantaneous variance σ^2
- Ito's Formula Short-hand:
  $$dX_t = \mu dt + \sigma dB_t$$
- Ruin Problem for X_t: calculate $P(X_\tau = A \mid X_0 = 0)$ with
  $\tau = inf\{t : X_t = A,\ or\ X_t = -B\}$ $(A > 0, B > 0)$
- Martingale Solution Approach: find function $h(\cdot)$:
  - $M_t = h(X_t)$ is a martingale; then M_τ is a martingale too.
  - Conditions on $h : h(A) = 1$ and $h(-B) = 0$.
  - $\Longrightarrow E[M_\tau] = E[h(X_\tau)] = P(X_\tau = A)$
  - By Martingale property: $P(X_\tau = A) = E[M_\tau] = M_0 = h(0)$.
- Apply the PDE Condition: $\frac{\partial h}{\partial t} = -\frac{1}{2}\frac{\partial^2 h}{\partial x^2}$

Example 5 (continued)

- $h(X_t) = h(\mu t + \sigma B_t)$
- So, work with $f(t, x) = h(\mu t + \sigma x)$
- $\frac{\partial f}{\partial t} = \mu h'(\mu t + \sigma x)$
- $\frac{\partial^2 f}{\partial x^2} = \sigma^2 h''(\mu t + \sigma x)$
- PDE Condition: $\frac{\partial f}{\partial t} = -\frac{1}{2} \frac{\partial^2 f}{\partial x^2}$

  $$\mu h'(\mu t + \sigma x) = -\frac{1}{2}\sigma^2 h''(\mu t + \sigma x)$$

  or

  $$h''(y) = -(2\mu/\sigma^2) h'(y),$$

  with $h(A) = 1$ and $h(-B) = 0$.
- Solution: $h(y) = \frac{exp(-2\mu y/\sigma^2) - exp(2\mu B/\sigma^2)}{exp(-2\mu A/\sigma^2) - exp(2\mu B/\sigma^2)}$

  $$\Longrightarrow P(X_\tau = A) = h(0).$$

Ito's Formula: General Case

Definition: A process $\{X_t, 0 \le t \le T\}$ is **standard** if $\{X_t\}$ has the integral representation

$$X_t = x_0 + \int_0^t a(\omega, s)ds + \int_0^T b(\omega, s)dB_s \text{ for } 0 \le t \le T$$

- $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are adapted, measurable with respect to $\{\mathcal{F}_t\}$
- $P(\int_0^t |a(\omega, s)|ds < \infty) = 1$
- $P(\int_0^t |b(\omega, s)|^2 ds < \infty) = 1$

Theorem (Ito's Formula for Standard Processes) If $f \in C^{1,2}(R^+ \times R)$ and $\{X_t, 0 \le t \le T\}$ is a **standard process** Then

$$\begin{aligned} f(t, X_t) &= f(0,0) + \int_0^t \frac{\partial f}{\partial t}(s, X_s)ds \\ &\quad + \int_0^t \frac{\partial f}{\partial x}(s, X_s)dX_s + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(s, X_s)b^2(\omega, s)ds \end{aligned}$$

Shorthand for $Y_t = f(t, X_t)$

$$dY_t = f_t(t, X_t)dt + f_x(t, X_t)dX_t + \tfrac{1}{2} f_{xx}(t, X_t)dX_t \cdot dX_t$$

Ito's Formula: Markov Case

Consider a generalized Wiener process $\{X_t, 0 \leq t \leq T\}$ with the integral representation

$$X_t = x_0 + \int_0^t \mu(X_s, s)ds + \int_0^T \sigma(X_s, s)dB_s \text{ for } 0 \leq t \leq T$$

or equivalently

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dB_t \text{ for } 0 \leq t \leq T$$

$\mu(\cdot, \cdot)$ and $\sigma(\cdot, \cdot)$ are adapted, measurable wrt $\{\mathcal{F}_t\}$

Ito's Lemma: Given a generalized Wiener process

$\{X_t, 0 \leq t \leq T\}$, consider the process $Y_t = G(X_t, t)$

where G is a smooth, differentiable function of X_t and t. Then

$$dG = \left[\frac{\partial G}{\partial x}\mu(X_t, t) + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}\sigma^2(X_t, t)\right] dt + \frac{\partial G}{\partial x}\sigma(X_t, t)dB_t$$

Ito's Formula: Markov Case

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Example 2 (again) $\mu(X_t, t) = 0$, $\sigma(X_t, t) = 1$, $G(x, t) = x^2$.

$\frac{\partial G}{\partial x} = 2x$, $\frac{\partial G}{\partial t} = 0$, $\frac{\partial^2 G}{\partial x^2} = 2$.

$$\implies d[X_t^2] = (2X_t \times 0 + 0 + \tfrac{1}{2} \times 2 \times 1)dt + 2X_t dX_t = dt + 2X_t dX_t.$$

Ito's Formula: Markov Case

Consider a generalized Wiener process $\{X_t, 0 \leq t \leq T\}$ with the integral representation

$$X_t = x_0 + \int_0^t \mu(X_s, s)ds + \int_0^T \sigma(X_s, s)dB_s \text{ for } 0 \leq t \leq T$$

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$$dG = \left[\frac{\partial G}{\partial x}\mu(X_t, t) + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}\sigma^2(X_t, t)\right] dt + \frac{\partial G}{\partial x}\sigma(X_t, t)dB_t$$

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$$\Longrightarrow d[X_t^2] = (2X_t \times 0 + 0 + \frac{1}{2} \times 2 \times 1)dt + 2X_t dX_t = dt + 2X_t dX_t.$$

So $X_t^2 = \int_0^t ds + 2\int_0^t X_s dX_s \iff \int_0^t X_s dX_s = \frac{1}{2}[X_t^2 - t]$

Example 6. Geometric Brownian Motion: $\{P_t, 0 \leq t < \infty\}$ stochastic process of an asset price which is an Ito Process

$$dP_t = \mu P_t dt + \sigma P_t dB_t$$

where μ and $\sigma > 0$ are constant. P_t is generalized Wiener process, $X_t = P_t$ with parameters: $\mu(x_t, t) = \mu x_t$, $\sigma(x_t, t) = \sigma x_t$

Ito's Formula: Markov Case

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Apply Ito's Lemma to obtain model for $G(P_t, t) = \ln(P_t)$

- $\frac{\partial G}{\partial P_t} = \frac{1}{P_t}$, $\frac{\partial G}{\partial t} = 0$, $\frac{1}{2}\frac{\partial^2 G}{\partial P_t^2} = \frac{1}{2}\frac{(-1)}{P_t^2}$.

Ito's Formula: Markov Case

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- Ito's Lemma gives

$$\begin{aligned} d\ln(P_t) &= \left(\frac{1}{P_t}\mu P_t + \frac{1}{2}\frac{(-1)}{P_t^2}\sigma^2 P_t^2\right) dt + \frac{1}{P_t}\sigma P_t dB_t \\ &= \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma dB_t \end{aligned}$$

Log price follows generalized Wiener process:

drift rate $=(\mu - \sigma^2/2)$ and variance rate $= \sigma^2$

Ito's Formula: Markov Case

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stochastic process of an asset price which is an Ito Process

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- $\frac{\partial G}{\partial P_t} = \frac{1}{P_t}$, $\frac{\partial G}{\partial t} = 0$, $\frac{1}{2}\frac{\partial^2 G}{\partial P_t^2} = \frac{1}{2}\frac{(-1)}{P_t^2}$.
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Log price follows generalized Wiener process:

drift rate $=(\mu - \sigma^2/2)$ and variance rate $= \sigma^2$

Log return on $(t, t + \Delta t]$: $\ln(\frac{P_{t+\Delta t}}{P_t}) \sim N([\mu - \frac{\sigma^2}{2}](\Delta t), \sigma^2 \Delta t))$.

Geometric Brownian Motion (continued)

Log return on $(t, t + \Delta t]$: $\ln(\frac{P_{t+\Delta t}}{P_t}) \sim N([\mu - \frac{\sigma^2}{2}](\Delta t), \sigma^2 \Delta t)$.

Geometric Brownian Motion (continued)

Log return on $(t, t + \Delta t]$: $\ln(\frac{P_{t+\Delta t}}{P_t}) \sim N([\mu - \frac{\sigma^2}{2}](\Delta t), \sigma^2 \Delta t))$.

Change in log price on the increment $(t, T]$

$$ln(P_T) - ln(P_t) \sim N([(\mu - \frac{\sigma^2}{2})(T - t), \sigma^2(T - t))$$

(sum of independent Normal increments)

Geometric Brownian Motion (continued)

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Change in log price on the increment $(t, T]$

$$ln(P_T) - ln(P_t) \sim N([(\mu - \frac{\sigma^2}{2})(T-t), \sigma^2(T-t))$$

(sum of independent Normal increments)

Conditional distribution of $ln(P_T)$ **given** P_t

$$ln(P_T) \sim N(ln(P_t) + (\mu - \frac{\sigma^2}{2})(T-t), \;\; \sigma^2(T-t))$$

$\Longrightarrow P_T$ is $LogNormal(\mu_*, \sigma_*^2)$ where

$$\begin{aligned} \mu_* &= ln(P_t) + (\mu - \frac{\sigma^2}{2})(T-t) \\ \sigma_*^2 &= \sigma^2(T-t) \end{aligned}$$

$$\begin{aligned} E[P_T \mid P_t] &= P_t e^{\mu_* + \frac{1}{2}\sigma_*^2} = P_t e^{(\mu - \frac{\sigma^2}{2})(T-t) + \frac{1}{2}\sigma^2(T-t)} = P_t^{\mu(T-t)} \\ Var[P_t \mid P_t] &= P_t^2 e^{2\mu(T-t)} \times [e^{\sigma^2(T-t)} - 1] \end{aligned}$$

Geometric Brownian Motion (continued)

$$
\begin{aligned}
E[P_T \mid P_t] &= P_t e^{\mu(T-t)} \\
Var[P_t \mid P_t] &= P_t^2 e^{2\mu(T-t)} \times [e^{\sigma^2(T-t)} - 1]
\end{aligned}
$$

- μ is the expected rate of return on the asset.
- Consider annual time scale ($T = 1$ year) with $\mu =$0.3 and $\sigma =$0.4 and $T - t =$0.5 (one-half year).

$$
\begin{aligned}
E[P_T \mid P_t] &= P_t e^{0.30\times 0.5} = P_t \times (1.161834) \\
Var[P_T \mid P_t] &= P_t^2[exp[2 \times .30 \times .5]][exp(.40^2(.5)) - 1.] \\
&= (P_t \times 0.3353)^2
\end{aligned}
$$

Geometric Brownian Motion (continued)

$$\begin{aligned} E[P_T \mid P_t] &= P_t e^{\mu(T-t)} \\ Var[P_t \mid P_t] &= P_t^2 e^{2\mu(T-t)} \times [e^{\sigma^2(T-t)} - 1] \end{aligned}$$

- μ is the expected rate of return on the asset.
- Consider annual time scale ($T = 1$ year) with $\mu =$0.3 and $\sigma =$0.4 and $T - t =$0.5 (one-half year).

$$\begin{aligned} E[P_T \mid P_t] &= P_t e^{0.30 \times 0.5} = P_t \times (1.161834) \\ Var[P_T \mid P_t] &= P_t^2 [exp[2 \times .30 \times .5]][exp(.40^2(.5)) - 1.] \\ &= (P_t \times 0.3353)^2 \end{aligned}$$

Continuously Compounded Rate of Return: r

$$P_T = P_t exp[r(T-t)] \Longrightarrow r = \frac{1}{T-t} \ln(\frac{P_T}{P_t})$$

Distribution for r:

$$r \sim N(\mu - \frac{1}{2}\sigma^2, \frac{\sigma^2}{T-t})$$

Geometric Brownian Motion (continued)

Simulating Brownian Motion (pdf)

- $\{W(t)\}$: Brownian motion model with drift $\mu \in R$ and volatility $\sigma > 0$.
    - $T = 4$(years)
    - $\mu = 0.30$, $\sigma = 0.40$
    - $m =$ increments on time interval $(0, T]$. Endpoints at $t_i = i \times (T/m)$, $i = 1, 2, \ldots, m$
- Geometric Brownian Motion (GBM): $X(t) = e^{W(t)}$
- Log Returns of GBM: $R(t) = log[X(t)/X(0)]/t$

Follow-Up Topics

Stochastic Differential Equations

- Derivatives Pricing Models
  - Martingales and Arbitrage Pricing
- Population Growth Models
- Existence and Uniqueness Theorems

Filtering Problems

- Innovation Processes
- Kalman-Bucy Filter

Optimal Stopping Problems

- Maximizing Expected Utility/Reward

Stochastic Control

18.642 Topics in Mathematics with Applications in Finance

Fall 2024