**Simulations of**
**Brownian Motion:** $W(t)$
**Geometric Brownian Motion (GBM):** $X(t) = e^{W(t)}$
**Log Returns of GBM:** $R(t) = log[X(t)/X(0)]/t$

Suppose $\{W(t)\}$ is a Brownian motion model with drift $\mu \in R$ and volatility $\sigma > 0$.

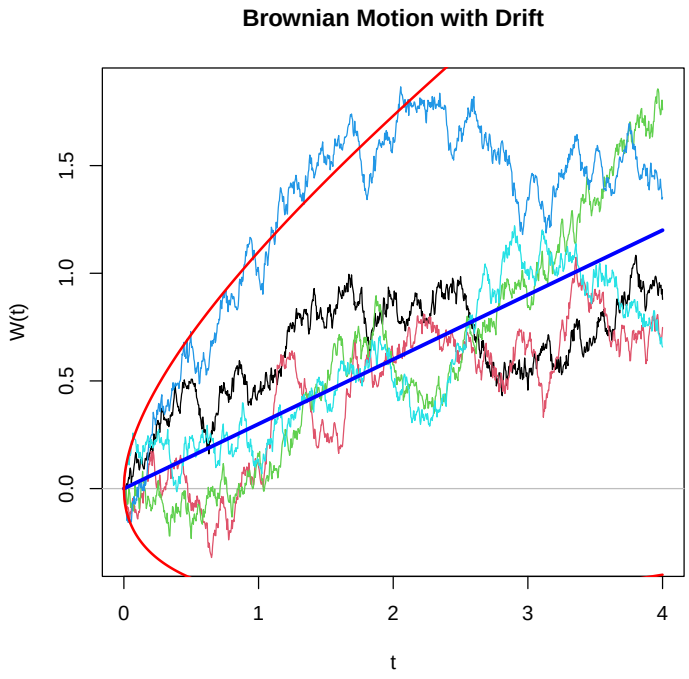
(a) Simulate the path of a Brownian Motion Process with the following properties:

- $T = 4$(years)
- $\mu = 0.30$, $\sigma = 0.40$
- $m =$ increments on time interval $(0, T]$. Endpoints at $t_i = i \times (T/m)$, $i = 1, 2, \ldots, m$
- Plot five simulated paths of the process (plotting just the process values at the time increments).

**Solution:**

```
> mu=.30 # annualized drift
> sigma=0.40 # annualized volatility
> T=4
> # Model increments with 252 per year
> m = T* 252 # number of increments
> mu.increment=T*mu/m
> sigma.increment = sigma*sqrt(T/m)
> #
> set.seed(1) # initialize random number generator seed
> t.grid<-seq(0,T,T/m)
> # Initialize matrix  Wmat.grid with (m+1) rows and 5 columns
> #   Each column will contain simulated path
>
> npaths=5
> Wmat.grid<-matrix(NA,nrow=m+1, ncol=npaths)
> for (j in c(1:npaths)){
+   deltaW.grid<-rnorm(m, mean=mu.increment,sd=sigma.increment)
+   W.grid<-c(0,cumsum(deltaW.grid))
+   Wmat.grid[,j]<-W.grid
+ }
> ylim0<-c(min(as.vector(Wmat.grid)), max(as.vector(Wmat.grid)))
> plot(x=t.grid, y=Wmat.grid[,1], ylim=ylim0, type="l", xlab="t", ylab="W(t)", main="Br
> abline(h=0,col='gray')
> for (j in 2:npaths){
+     lines(x=t.grid,y=Wmat.grid[,j], col=j)}
> #
```

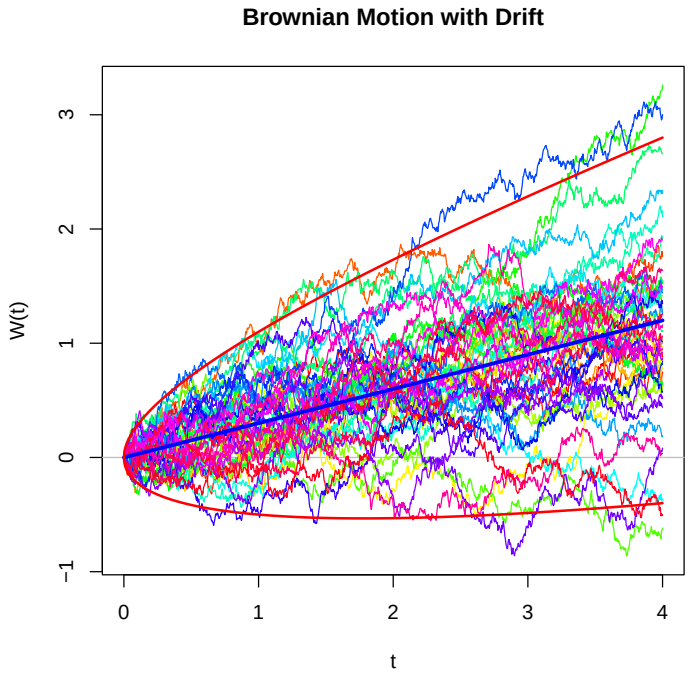
```
> # Add lines for mean level of process and plus/minus 2 sds as functions of time t
>
> mu.grid=t.grid*mu
> sigma.grid<-sqrt(t.grid)*sigma
> lines(x=t.grid,y=mu.grid, col='blue',lwd=3)
> lines(x=t.grid,y=mu.grid + 2*sigma.grid,col='red',lwd=2)
> lines(x=t.grid,y=mu.grid -2*sigma.grid,col='red',lwd=2)
```




(b) Repeat with 50 paths

```
> # Repeat with 50 paths
> npaths=50
> set.seed(1)
> Wmat.grid<-matrix(NA,nrow=m+1, ncol=npaths)
> for (j in c(1:npaths)){
+   deltaW.grid<-rnorm(m, mean=mu.increment,sd=sigma.increment)
+   W.grid<-c(0,cumsum(deltaW.grid))
+   Wmat.grid[,j]<-W.grid
+ }
> ylim0<-c(min(as.vector(Wmat.grid)), max(as.vector(Wmat.grid)))
> colset<-rainbow(npaths)
> plot(x=t.grid, y=Wmat.grid[,1], ylim=ylim0, type="l", col=colset[1], main="Brownian M
```

```
+      ylab="W(t)", xlab="t")
> abline(h=0,col='gray')
> for (j in 2:npaths){
+     lines(x=t.grid,y=Wmat.grid[,j], col=colset[j])}
> #
> # Add lines for mean level of process and plus/minus 2 sds as functions of time t
>
> mu.grid=t.grid*mu
> sigma.grid<-sqrt(t.grid)*sigma
> lines(x=t.grid,y=mu.grid, col='blue',lwd=3)
> lines(x=t.grid,y=mu.grid + 2*sigma.grid,col='red',lwd=2)
> lines(x=t.grid,y=mu.grid -2*sigma.grid,col='red',lwd=2)
```

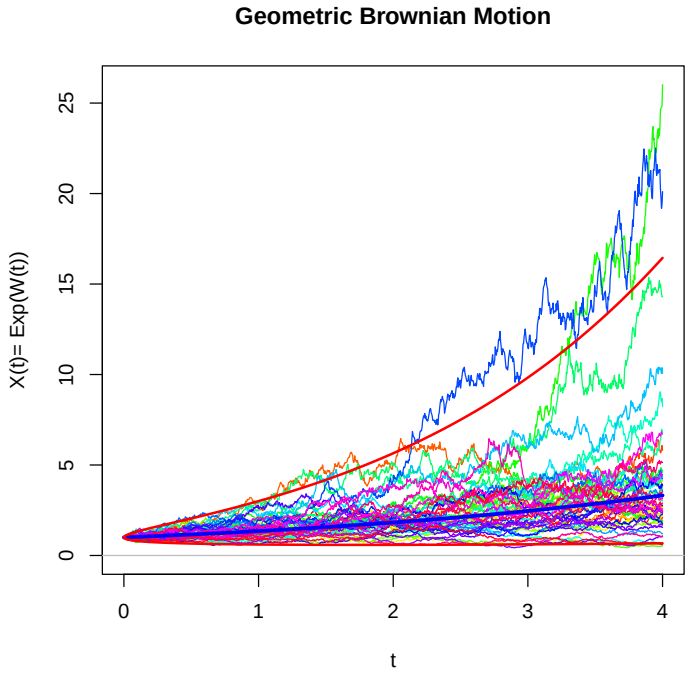



(c) Plot Geometric Brownian Motion with log process given by $W$ process in prior part:

$x(t) = exp(W(t))$.

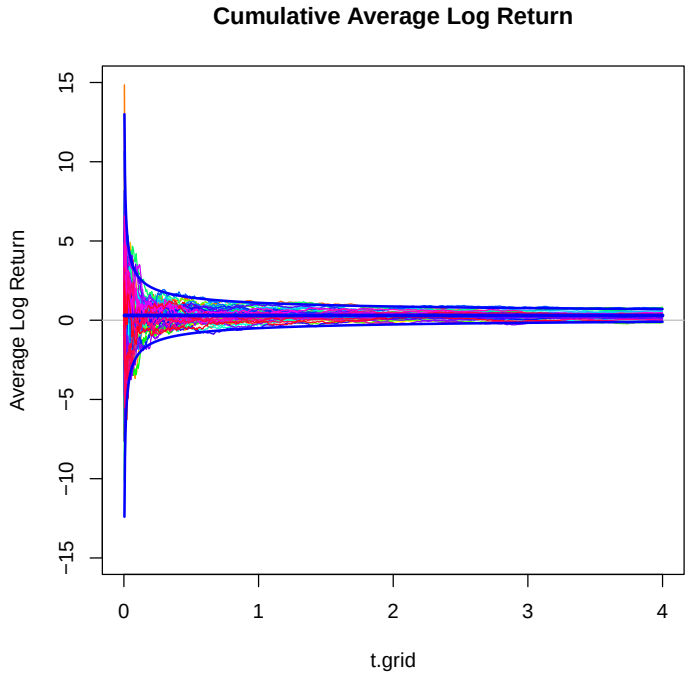
```
> Xmat.grid<-exp(Wmat.grid)
> Xmat.maxabs<-max(as.vector(abs(Xmat.grid)))
> ylim00<- c(0, Xmat.maxabs)
> colset<-rainbow(npaths)
> plot(x=t.grid, y=Xmat.grid[,1], ylim=ylim00, type="l", col=colset[1],
```

```
+      main="Geometric Brownian Motion",
+      ylab="X(t)= Exp(W(t))",
+       xlab="t")
> abline(h=0,col='gray')
> for (j in 2:npaths){
+     lines(x=t.grid,y=Xmat.grid[,j], col=colset[j])}
> #
> # Add lines for exponent of mean level of process
> # Add lines for exponent of mean level of process and plus/minus 2 sds as functions o
>
> mu.grid=t.grid*mu
> sigma.grid<-sqrt(t.grid)*sigma
> lines(x=t.grid,y=exp(mu.grid), col='blue',lwd=3)
> lines(x=t.grid,y=exp(mu.grid + 2*sigma.grid),col='red',lwd=2)
> lines(x=t.grid,y=exp(mu.grid -2*sigma.grid)
+       ,col='red',lwd=2)
>
```



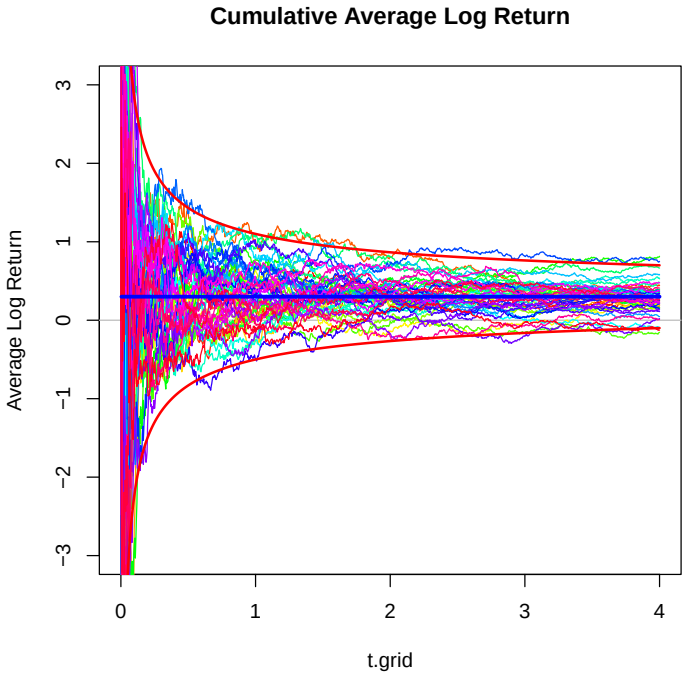

(d) Plot Cumulated Average Log Return for Geometric Brownian Motion with log process given by $W$ process in prior part:
$R(t) = \frac{log(X(t)/X(0))}{t} = W(t)/t.$

```
> # Plot Returns to time t
> # for same 50 paths
>
> Rmat.grid<-Wmat.grid
> for (j in c(1:ncol(Wmat.grid))){
+   Rmat.grid[,j]<- Wmat.grid[,j]/t.grid
+ }
> Rmat.grid[1,]<-0
> Rmat.maxabs<- max(as.vector(abs(na.omit(Rmat.grid))))
> ylim00<- c(-Rmat.maxabs, Rmat.maxabs)
> colset<-rainbow(npaths)
> plot(x=t.grid, y=Rmat.grid[,1], ylim=ylim00, type="l", col=colset[1],
+      main="Cumulative Average Log Return",
+      ylab="Average Log Return")
> abline(h=0,col='gray')
> for (j in 2:npaths){
+     lines(x=t.grid,y=Rmat.grid[,j], col=colset[j])}
> #
> # Add lines for mean level of process and plus/minus 2 sds
>
> mu.grid=t.grid*mu/t.grid
> sigma.grid<-sqrt(t.grid)*sigma/t.grid
> lines(x=t.grid,y=mu.grid, col='blue',lwd=3)
> lines(x=t.grid,y=mu.grid + 2*sigma.grid,col='blue',lwd=2)
> lines(x=t.grid,y=mu.grid -2*sigma.grid,col='blue',lwd=2)
>
```




Repeat with y limits closer to long-run log return

```
> #ylim00<- c(-Rmat.maxabs, Rmat.maxabs)
> ylim00<-c(-3,3)
> colset<-rainbow(npaths)
> plot(x=t.grid, y=Rmat.grid[,1], ylim=ylim00, type="l", col=colset[1],
+      main="Cumulative Average Log Return",
+      ylab="Average Log Return")
> abline(h=0,col='gray')
> for (j in 2:npaths){
+     lines(x=t.grid,y=Rmat.grid[,j], col=colset[j])}
> #
> # Add lines for mean level of process and plus/minus 2 sds
>
> mu.grid=t.grid*mu/t.grid
> sigma.grid<-sqrt(t.grid)*sigma/t.grid
> lines(x=t.grid,y=mu.grid, col='blue',lwd=3)
> lines(x=t.grid,y=mu.grid + 2*sigma.grid,col='red',lwd=2)
> lines(x=t.grid,y=mu.grid -2*sigma.grid,col='red',lwd=2)
>
```


Cumulative Average Log Return
Average Log Return
3
2
1
0
−1
−2
−3
0
1
2
3
4
t.grid

# 18.642 Topics in Mathematics with Applications in Finance

Fall 2024