18.642 Assignment 1 Fall 2024

Collaboration on homework is encouraged, but you will benefit from independent effort to solve the problems before discussing them with other people. **You must write your solution in your own words. List all your collaborators.**

1. **Fibonacci Numbers and the Fibonacci Matrix**

   Strang (2006) calls the matrix

   $$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

   the "Fibonacci matrix." Consider the sequence of Fibonacci numbers

   $$\{0, 1, 1, 2, 3, 5, 8, 13, \ldots\} = \{F_0, F_1, F_2, \ldots\}$$

   These number satisfy $F_{k+2} = F_{k+1} + F_k$, for $k = 0, 1, 2, \ldots$

   Consider the 2-vector $\vec{u}_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$

   (a) Show that the sequence $\{\vec{u}_k, k = 0, 1, 2, \ldots\}$ satisfies $\vec{u}_{k+1} = A\vec{u}_k$.

   (b) Show that $\vec{u}_k = A^k \vec{u}_0$.

   (c) Solve for λ_1, λ_2, the two eigenvalues of A.
   Note the λs solve $det(A - \lambda \mathrm{I}_2) = 0$, where I_2 is the 2-by-2 identity matrix.

   (d) An eigenvector $\vec{s}$ of A corresponding to eigen value λ solves $A\vec{s} = \lambda\vec{s}$. Show that for eigenvalue λ, an eigenvector is given by

   $$\vec{s} = \begin{bmatrix} \lambda \\ 1 \end{bmatrix}$$

   (e) Define $\vec{s}_1 = \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix}$, $\vec{s}_2 = \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$ and the 2-by-2 matrix

   $$S = [\vec{s}_1 \; \vec{s}_2]$$

   with columns equal to the respective eigen-vectors.
   Compute S^{-1}, the inverse of matrix S.

   (f) Show that $A = S\Lambda S^{-1}$, where Λ is the diagonal matrix with $\Lambda_{ii} = \lambda_i$, $i = 1, 2$.

   (g) Compute the vector

   $$\vec{c} = S^{-1}\vec{u}_0.$$

   and simplify to show that $\vec{c} = a \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, for some constant a. What is a?

(h) Compute $A^k\vec{u}_0 = S\Lambda^k S^{-1}\vec{u}_0 = S\Lambda^k\vec{c}$. Show that the kth Fibonacci number is given by

$$F_k = \frac{1}{\sqrt{5}}\left[\lambda_2^k - \lambda_1^k\right]$$

where $\lambda_1 < \lambda_2$.

(i) What is the limiting ratio of F_{k+1} to F_k? This is the Golden-Ratio. This ratio occurs widely in nature as well as in the technical analysis of asset prices (e.g., price retracements and target price moves).

2. If an $m \times m$ matrix A has m eigenvalues λ_i (which can repeat) and m linearly independent eigen vectors $\vec{s}_i$, then A is *diagonalizable*:

$$A = S\Lambda S^{-1},$$

where the $m \times m$ matrix S has columns given by the eigen-vectors $\vec{s}_i$ and the $m\times m$ diagonal matrix Λ has diagonal elements equal to $\lambda_1, \lambda_2, \ldots, \lambda_m$, the eigenvalues of A.

Consider the limit of the sequence of matrices $\{A^k, k = 1, 2, \ldots\}$ given by powers of A.

(a) Under what conditions does the limit not exist (i.e., the limit of some elements of A^k diverge or do not converge)?

(b) Under what conditions does the limit exist as the zeroes matrix (all elements are 0)?

(c) Under what conditions does the limit exist as a finite matrix that is not all zeroes?

3. Let $\{X_t, t = 1, 2, \ldots\}$ be a stochastic process representing the state of the process at time t. If the process is Markov, then the conditional probability distribution of X_t given by $P[X_t \mid X_{t-1}, X_{t-2}, \ldots, X_0]$ satisfies:

$P[X_t \mid X_{t-1}, X_{t-2},], \ldots, X_0] = P[X_t \mid X_{t-1}]$, i.e.,

the distribution depends only on the last state X_{t-1} before t.

The process is called a two-state *Markov Chain*, with states 1 and 2, if the transition probability matrix for all times is given by the 2×2 matrix A:

$$A = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}$$

where $A_{i,j} = P[X_t = i \mid X_{t-1} = j]$, $i, j = 1, 2$.

Note that the columns of A are non-negative and sum to 1; these are properties of a *stochastic* matrix.

For $t = 0, 1, 2, \ldots$, define $\vec{u}_t$, $t = 1, \ldots$ as follows:

$$\vec{u}_0 = \begin{bmatrix} P[X_0 = 1] \\ P[X_0 = 2] \end{bmatrix}, \text{ and } \vec{u}_t = \begin{bmatrix} P[X_t = 1 \mid \vec{u}_0] \\ P[X_t = 2 \mid \vec{u}_0] \end{bmatrix}.$$

For $t > 0$, $\vec{u}_t$ gives the conditional probability distribution of X_t, the state at time t given the distribution of X_0.

An example of a two-state Markov chain is side (Buy or Sell) of successive market orders for an asset in a stationary, efficient market, where Buy orders and Sell orders arrive at the market at different (but fixed) rates depending on the side of the last order. Suppose state 1 corresponds to a Buy order and state 2 corresponds to a Sell order. Under the Markov Chain model, the probability of a Buy order following a Buy order is 0.8 and the probability of a Buy order following a Sell order is 0.3. Also, the probability of a Sell order following a Sell order is 0.7 (smaller than for Buy following Buy), and the probabilty of a Sell order following a Buy order is 0.2.

(a) Prove that the $\vec{u}_t$ satisfy:

$$\vec{u}_t = A\vec{u}_{t-1}$$

(b) Prove that the $\vec{u}_t$ satisfy:

$$\vec{u}_t = A^k\vec{u}_{t-k}$$

for $k = 1, 2, \ldots$; and in particular $u_t = A^t\vec{u}_0$.

(c) If $X_0 = 1$, i.e., $\vec{u}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, the probability distribution of X_t, (i.e., $\vec{u}_t$), for $t = 1, 2, \ldots, 5$ can be computed numerically using R:

```
> # Define the matrix A and the column matrix u0
> A=matrix(c(.8,.3,.2,.7),byrow=TRUE, ncol=2)
> u0= as.matrix(c(1,0))
> # Compute u1 = A u0
> u1 = A %*% u0
> u1

     [,1]
[1,]  0.8
[2,]  0.2

> # Compute u2 = A u1
> u2=A %*% u1
> u2

     [,1]
[1,]  0.7
[2,]  0.3

> # Use a loop to compute ut, for t=1,2,...,tmax
> ut=u0
> tmax=5
> for (t in c(1:tmax)){
+   ut=A %*% ut
+   if (t <= tmax){
+   cat("t= ",t,":\n")
+   print(ut)
+   }
+ }
```

```
t=  1 :
     [,1]
[1,]  0.8
[2,]  0.2
t=  2 :
     [,1]
[1,]  0.7
[2,]  0.3
t=  3 :
     [,1]
[1,] 0.65
[2,] 0.35
t=  4 :
      [,1]
[1,] 0.625
[2,] 0.375
t=  5 :
       [,1]
[1,] 0.6125
[2,] 0.3875
```

Increase the upper limit $tmax$ on t to determine when $\vec{u}_t$ converges (numerically in R), i.e., $\vec{u}_* = \vec{u}_{tmax}$. Reset the value of $tmax$ and change the 'if' condition to $(t > tmax - 3)$ to demonstrate convergence.

(d) Repeat part (c) if $X_0 = 2$, i.e., $\vec{u}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Does the (numerical) limit $\vec{u}_*$ of $\vec{u}_t$ converge to the same limit as part (c)?

(e) The limiting vector $\vec{u}_*$ in part (c) above is an eigen-vector of the matrix A corresponding to the eigen-value $\lambda = 1$. It corresponds to the *stationary* distribution – the distribution of X_t for which the distributions of X_{t+k} do not change for $k > 0$. In the example of the Markov Chain for side of successive market orders in a stationary market, what is the stationary distribution of order side?

The eigen-values/eigen vectors of A can be computed in R:

```
> A.eigen=eigen(A)
> print(A.eigen)
```

```
eigen() decomposition
$values
[1] 1.0 0.5

$vectors
          [,1]       [,2]
[1,] 0.8320503 -0.7071068
[2,] 0.5547002  0.7071068
```

```
> A.eigen.value1=A.eigen$value[1]
> A.eigen.vector1=as.matrix(A.eigen$vectors[,1])
> A.eigen.value1
[1] 1
> A.eigen.vector1
          [,1]
[1,] 0.8320503
[2,] 0.5547002
> # Demonstrate the property of eigenvector1
> A %*% A.eigen.vector1
          [,1]
[1,] 0.8320503
[2,] 0.5547002
> A.eigen.value1 * A.eigen.vector1
          [,1]
[1,] 0.8320503
[2,] 0.5547002
> # These are equal
```

Explain the relationship between the limiting vector $\vec{u}_*$, an eigenvector of A and the output of the R function $eigen()$.

4. For the matrix A in problem 3

$$A = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}$$

   (a) Solve explicitly for the eigenvalues of A.

   (b) Explain why case (c) of problem 2 applies for the matrix A.

   (c) Explain the connection between the stationary distribution in part (e) of Problem 3 and the Perron-Frobenius Theorem.

5. In the general case of the One-Period Economy with Two Assets (see lecture note "One-Period Financial Models"), prove that the hypothesis of no arbitrage is satisfied only if the following strict inequality is satisfied:

$$\frac{S_T^d}{1 + r_F T} < S_0 < \frac{S_T^u}{1 + r_f T}.$$

equivalently

$$S_T^d < S_0 \times (1 + r_F T) < S_T^u.$$

Hint: Consider a violation of either inequality and construct a portfolio and trading strategy with arbitrage (i.e., its cost C_0 at $t = 0$ is lower than its payoffs at $t = T$).