18.642 Problem Set 3 Fall 2024

Collaboration on homework is encouraged, but you will benefit from independent effort to solve the problems before discussing them with other people. **You must write your solution in your own words. List all your collaborators.**

Problem 1. Prove Stein's Lemma:

- Suppose that X and Y are jointly normally distributed with $X \sim Normal(\mu, \sigma^2)$.
- Let g be a continuous, differentiable function for which

  $$E[g(X)(X-\mu)] \text{ and } E[g'(X)] \text{ exist.}$$

Then:

$$\begin{aligned} E[g(X)(X-\mu)] &= \sigma^2 E[g'(X)] \text{ and} \\ Cov[g(X), Y] &= E[g'(X)]Cov(X, Y). \end{aligned}$$

Hint: Consider integration by parts. Also, you can use/assume the fact that if X and Y are jointly normally distributed (i.e., bivariate normal distribution) with mean vector and covariance matrix

$$E\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} \mu_X \\ \mu_Y \end{bmatrix} \text{ and } Cov\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} \sigma^2 & \rho\sigma\tau \\ \rho\sigma\tau & \tau^2 \end{bmatrix}$$

Then the random variable Y can be written as $Y = a + bX + \epsilon$,
where $a = \mu_Y - b\mu_X$ and $b = \rho\tau/\sigma$ and ϵ is a random variable independent of X with distributon $N(0, (1-\rho^2)\tau^2)$.

Problem 2. Stein's Lemma for Stochastic Volatility

Gron, Jorgensen and Polson (2011) extend Stein's lemma to the case where the asset return (X) has stochastic volatility. They state:

> In general, a random variable X whose volatility is drawn from a probability density function $p(V)$ is said to exhibit stochastic volatility, V, if we can write $X \mid V \sim N(\mu, \sigma^2 V)$, where $V \sim p(V)$, $V > 0$, and $\sigma^2 > 0$.
>
> The distribution of outcomes X is
>
> $$p(X) = \int_0^\infty p(X \mid V)p(V)dV$$
>
> With stochastic volatility, the marginal distribution of X has heavy tails. Stein's lemma equates the covariance of a function of normal random variables to the underlying covariance times a proportionality factor. The critial difference between the Stein's lemma and our derivation is that the proportionality factor undergoes a change of measure that we denote by $q(V)$. This density comes from size-biasing the density of the volatility and is defined as
>
> $$q(V) = \frac{Vp(V)}{E[V]},$$
>
> where we assume that $0 < E[V] < \infty$. We define $q(X)$ to be the induced marginal distributino of X given by this measure $q(V)$ on the volatlity, namely
>
> $$q(X) = \int_0^\infty p(X \mid V)q(V)dV.$$
>
> In general, size-biasing in this way casues the density $q(X)$ to have heavier tails than the orignal density $p(X)$.
>
> *Theorem 1 (Stein's lemma for stochastic volatility)*
>
> Let X be a random variable with stochastic volatility so that $X \mid V$ is distributed $N(\mu, \sigma^2 V)$ and V has density $p(V)$, that is non-negative only vor $V \geq 0$. Suppose that $0 < E[V] < \infty$. Then we have
>
> $$Cov[g(X), X] = E^Q[g'(X)]Var[X], \quad (**)$$
>
> where E^Q is the expectation taken under the measure induced by size-biasing $q(V) = Vp(V)/E[V]$.
>
> ... (see the paper for further development)

(a) Show that the function $q(V)$, is a proper probability density on $\{V : 0 < V < \infty\}$.

In their proof of the theorem (Appendix A), they consider $Cov[g(X), X]$ and argue:

$$\begin{aligned} Cov[g(X), X] &= E[g(X)(X - E[X])] & (1) \\ &= E_V\{E_{X|V}[g(X)(X - E[X])]\} & (2) \\ &= E_V\{E_{X|V}[g(X)(X - E[X \mid V])]\} & (3) \end{aligned}$$

(b) Prove (1).

(c) Does (2) follow by the *Law of Iterated Expectations*?

(d) Prove step (3).

Hint: Compare $E[X \mid V]$ and $E[X]$

Their proof continues: apply Stein's lemma conditionally ($X \mid V$ is normally distributed as $N(\mu, \sigma^2 V)$) to obtain:

$$\begin{aligned} Cov[g(X), X] &= E_V\{E_{X|V}[g'(X)]Var[X \mid V]\} \\ &= E_V\{E_{X|V}[g'(X)]\sigma^2 V\} \\ &= E\{g'(X)\tfrac{V}{E[V]} \ \ \sigma^2 E[V] \ \ (*) \end{aligned}$$

(e) Does $(*)$ result imply $(**)$ in the theorem? Explain.

Note: Stochastic volatility is one of many possible final paper topics. Gron, Jorgensen and Polson (2011) examine the effect of stochastic volatility on optimal portfolio choice in equilibrium settings, detailing connections to classic results (Samuelson-Merton optimal portfolio), and its impact on risk aversion and asset prices in equilibrium. The paper cites additional references that may stimulate interest.

Problem 3. Stochastic Process for Asset Price Dynamics. Let $\{P_t, t = 0, 1, \ldots\}$ be the discrete-time stochastic process for an asset price. Consider a random-walk model for the logarithm of the asset price:

$$S_t = log(P_t)$$

Let $\{X_t, t = 0, 1, 2, \ldots\}$ be the stochastic process for the random-walk steps

$$X_t = S_t - S_{t-1} = log(P_t/P_{t-1})$$

and assume that the X_t are independent, identically distributed with

$$\begin{aligned} E[X_t] &= \mu, \ \ with \ \ -\infty < \mu < \infty \\ Var[X_t] &= \sigma^2, \ \ with \ \ \sigma^2 > 0. \end{aligned}$$

(a) For what parameter values (μ, σ^2) is the process $\{S_t\}$ a Markov Process? (Explain your reasoning).

(b) For what parameter values (μ, σ^2) is the process $\{S_t\}$ a Martingale? (Explain your reasoning).

(c) If $S_0 = 0$, consider the probability that the log price rises by more than 0.20 before it falls by more than $-.20$.

Let $\tau = min\{t : S_t > 0.20, \ \ or \ \ S_t < -0.20\}$, a *stopping time.* If S_t are i.i.d. $Normal(\mu, \sigma^2)$, what conditions on μ, and σ^2 make

$$P[S_\tau > 0.20] > \tfrac{1}{2}?$$

Problem 4. Principal Components Analysis of Yields

Consider the daily yield rate data for constant maturity US Treasury securities, taken from the Federal Reserve Economic Database (FRED); see case study on principal components analysis. To analyze changes in yields, the daily changes in yield, in basis point units; 1 basis point (BP) $= 0.01 \times 1\%$ were computed for the 9 securities.

Two principal components analyses are conducted on the data for 2001-2005, the first using the sample covariance matrix and the second using the sample correlation matrix. The results are:

US Tresaury Yield Data: 2001-2005

Principal Components Analysis of Yield Changes Covariance Matrix

Importance of components:

	Comp.1	Comp.2	Comp.3	Comp.4	Comp.5
Standard deviation	0.1618033	0.05323436	0.02988340	0.01918045	0.013461432
Proportion of Variance	0.8494434	0.09194833	0.02897475	0.01193650	0.005879525
Cumulative Proportion	0.8494434	0.94139169	0.97036644	0.98230294	0.988182464

	Comp.6	Comp.7	Comp.8	Comp.9
Standard deviation	0.011196400	0.009568498	0.009009979	0.008131889
Proportion of Variance	0.004067397	0.002970621	0.002633948	0.002145570
Cumulative Proportion	0.992249861	0.995220482	0.997854430	1.000000000

	Comp.1	Comp.2	Comp.3	Comp.4	Comp.5	Comp.6	Comp.7	Comp.8	Comp.9
DGS3MO	0.1088	0.5225	0.5434	0.5408	0.2107	0.2865	0.0041	0.0093	0.0291
DGS6MO	0.1581	0.4817	0.2508	-0.2555	-0.2581	-0.7343	-0.0634	-0.0473	-0.0524
DGS1	0.2531	0.4107	-0.0514	-0.6687	-0.0306	0.5244	0.1674	0.0914	0.0712
DGS2	0.3905	0.2071	-0.4750	0.0673	0.4529	-0.0503	-0.5029	-0.2849	-0.1809
DGS3	0.4197	0.0873	-0.4022	0.2965	0.0657	-0.2045	0.5551	0.4197	0.1900
DGS5	0.4206	-0.1082	-0.0311	0.2101	-0.5932	0.1841	0.1817	-0.4776	-0.3455
DGS7	0.4009	-0.2289	0.1475	0.0204	-0.2167	0.0356	-0.3865	-0.0047	0.7530
DGS10	0.3697	-0.2837	0.2710	-0.0666	-0.0173	0.0351	-0.2898	0.6191	-0.4859
DGS20	0.3101	-0.3621	0.3944	-0.2316	0.5286	-0.1515	0.3749	-0.3472	0.0033

Principal Components Analysis of Yield Changes Correlation Matrix

Importance of components:

	Comp.1	Comp.2	Comp.3	Comp.4	Comp.5
Standard deviation	2.6338307	1.1892266	0.56520358	0.39707758	0.26938548
Proportion of Variance	0.7707849	0.1571400	0.03549501	0.01751896	0.00806317
Cumulative Proportion	0.7707849	0.9279249	0.96341989	0.98093885	0.98900202

	Comp.6	Comp.7	Comp.8	Comp.9
Standard deviation	0.207981232	0.148493832	0.13664433	0.122488906
Proportion of Variance	0.004806244	0.002450046	0.00207463	0.001667059
Cumulative Proportion	0.993808264	0.996258311	0.99833294	1.000000000

	Comp.1	Comp.2	Comp.3	Comp.4	Comp.5	Comp.6	Comp.7	Comp.8	Comp.9
DGS3MO	-0.2083	0.6293	0.5831	0.4102	0.2282	0.0051	0.0061	0.0055	0.0110
DGS6MO	-0.2830	0.5154	-0.0654	-0.4899	-0.6291	-0.1177	0.0065	0.0034	-0.0155
DGS1	-0.3369	0.2698	-0.4066	-0.3784	0.6637	0.2334	-0.0682	-0.0616	0.0426
DGS2	-0.3625	0.0064	-0.4005	0.3598	0.0294	-0.4715	0.3772	0.4081	-0.2162
DGS3	-0.3664	-0.0665	-0.2719	0.3997	-0.1612	-0.1350	-0.3427	-0.6206	0.2845
DGS5	-0.3670	-0.1619	0.0036	0.1720	-0.1919	0.5277	-0.4297	0.3013	-0.4676
DGS7	-0.3608	-0.2298	0.1435	-0.0112	-0.0982	0.2782	0.2211	0.3472	0.7347
DGS10	-0.3525	-0.2656	0.2554	-0.1255	-0.0087	0.1746	0.5990	-0.4745	-0.3313
DGS20	-0.3290	-0.3338	0.4125	-0.3376	0.1955	-0.5521	-0.3784	0.0895	-0.0450

4(a) Compare the loadings (eigen-vectors) of the first principal component variable for the two cases. The loadings are all positive for the covariance case and all negative for the correlation case. Is the difference in sign meaningful? (Hint: consider the eigen-vector/value decomposition of the matrices; does the decomposition change if any eigen-vector is multiplied by -1?)

4(b) The magnitudes of the loadings for the first principal component has a larger range from smallest to largest for the covariance matrix case compared to the correlation matrix case. The loading on the least variable yield change (DGS3MO) is higher in magnitude for the correlation matrix case. Also, the loadings on the highly variable yield changes are lower in magnitude for correlation matrix case.

Provide a logical explanation for why the range of magnitudes is larger for the covariance case. (Recall that the first principal component variable is the normalized weighted average of the yield-change variables which has the highest variance (the normalized weights have sum-of-squares equal to 1). For the covariance matrix case, the yield-change variables are the original variables while for the correlation matrix case, these yield-change variables have been scaled to have mean 0 and variance 1.)

4(c) Provide an interpretation of the first three principal component variables for the correlation matrix case. Compare these to an interpretation of those for the covariance matrix case.

4(d) If the analysis objective is to model dynamics of the term-structure of interest rates across all tenors, argue why the principal components analysis of the correlation matrix might be preferred to that of the covariance matrix.

4(e) Compute the Total Variance (sum of variances of principal components variables) for the correlation case of the PCA. Is this value an integer? If so, explain why.